1. Show, using the hook-length formula, that the number of SYT of shape a $2 \times n$ rectangle is equal to the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

2. Let $A_n$ denote the set of 0, 1-sequences with $n$ 0-s and $n$ 1-s, such that in any initial subsequence there are at least as many 1-s as 0-s. For example $A_3 = \{111000, 110100, 110010, 101100, 101010\}$. Give a bijection between $A_n$ and the set of SYT of shape a $2 \times n$ rectangle.

3. A partition $\lambda$ is self-conjugate if $\lambda = \lambda'$. Prove that the number of self-conjugate partitions of $n$ is equal to the number of partitions of $n$ into odd distinct parts.

4. Fix a partition $\lambda$. Find a simple formula for the sum $\sum_{\mu \succ \lambda} f^{\mu}$.

5. A skew shape is a collection $\lambda/\mu$ of boxes belonging to $\lambda$ but not to $\mu$, where $\mu \subset \lambda$. A skew shape is a $n$-ribbon if it contains $n$ boxes, is edgewise-connected, and does not contain any $2 \times 2$ box. For example, $(2,2)/(1)$ is a 3-ribbon but $(3,1)/(1)$ is not because it is not edgewise connected.

Let $\lambda$ be a partition. The $n$-core of $\lambda$ is the partition obtained from $\lambda$ by repeatedly removing $n$-ribbons until this is no longer possible. Show that the $n$-core is well-defined, i.e., it does not depend on how the $n$-ribbons are moved.

6. Define a poset $P_n$ on the set of all partitions by $\lambda \leq \mu$ if $\lambda$ can be obtained from $\mu$ by repeatedly removing $n$-ribbons. Define linear operators $U, D : \mathbb{R}P_n \to \mathbb{R}P_n$ by

$$U\lambda = \sum_{\mu \succ \lambda} \mu \quad \text{and} \quad D\lambda = \sum_{\mu \preceq \lambda} \mu.$$ 

Show that

$$DU - UD = n\text{Id}.$$ 

7. Fix a partition $\lambda$ of $n$. Recall the random process defined in class: pick a cell $(a, b)$ of $\lambda$ with uniform probability $1/n$, then pick one of the remaining cells of the hook $H_{ab}$ with uniform probability $1/(h_{ab} - 1)$ and repeat until we arrive at a corner cell $(c, d)$. Let $p((c, d)|(a, b))$ be the probability that we end up at corner cell $(c, d)$ given that the initial cell is $(a, b)$. Show that

$$p((c, d)|(a, b)) = p((c, d)|(a, d))p((c, d)|(c, b)).$$