Problem 1  Consider the Poset given by the following diagram

\begin{center}
\begin{tikzpicture}

\node (A1) at (2,3) {$A_1$};
\node (A2) at (4,3) {$A_2$};
\node (B1) at (1,1) {$B_1$};
\node (B2) at (3,1) {$B_2$};
\node (C1) at (1,-1) {$C_1$};
\node (C2) at (2,-1) {$C_2$};
\node (C3) at (3,-1) {$C_3$};
\node (C4) at (4,-1) {$C_4$};

\draw (A1) -- (B1);
\draw (A1) -- (B2);
\draw (A2) -- (B1);
\draw (A2) -- (B2);
\draw (B1) -- (C1);
\draw (B1) -- (C3);
\draw (B2) -- (C2);
\draw (B2) -- (C4);
\draw (C1) -- (C2);
\draw (C3) -- (C4);
\end{tikzpicture}
\end{center}

Note that this poset can clearly be checked to be a graded poset and that it is Sperner.
Now consider $G = \mathbb{Z}/2\mathbb{Z}$ acting by swapping $A_i$’s and $B_i$’s and fixing everything else. This is clearly a poset automorphism. The quotient poset is the following

\begin{center}
\begin{tikzpicture}

\node (C1) at (1,-1) {$C_1$};
\node (C2) at (2,-1) {$C_2$};
\node (C3) at (3,-1) {$C_3$};
\node (C4) at (4,-1) {$C_4$};

\draw (C1) -- (C2);
\draw (C3) -- (C4);
\end{tikzpicture}
\end{center}
Now the $C_i$'s form an antichain of size 4, but the maximal size of a fixed rank is size 3, thus this is not Sperner as required.

**Problem 2** We will consider the poset $B_n/G$. By the results proven in class, this is a rank-symmetric and rank-unimodal poset. Now being $i$-transitive is equivalent to $B_n/G$ containing a unique element. Now suppose this is true for $i \leq \lfloor \frac{n}{2} \rfloor$ and let $j \leq i$. Then let $p_k = \sharp(B_n/G)_k$. By rank-symmetry we know that $p_{n-j} = p_j$. So if $p_j > 1$ we would have $p_j > p_i < p_{n-j}$ contradicting that this poset is rank-unimodal. Thus we must have $p_j = 1$ or in other words, $G$ is $j$-invariant as required.

**Problem 3** We will describe $B_M$ as a quotient $B_n/G$ for some group $G \subset \mathfrak{S}_n$ for $n = \sharp M$.

We start by introducing some notation. Denote by $A = (1^{k_1}, 2^{k_2}, \ldots)$ the multiset with 1 appearing $k_1$ times, 2 appearing $k_2$ times, etc. Now assume $M = (1^{n_1}, 2^{n_2}, \ldots)$. We then let $G = \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \cdots \subset \mathfrak{S}_n$.

We give an isomorphism $B_M \cong B_n/G$.

We define the first map

$$B_M \to B_n/G$$

$$A = (1^{k_1}, 2^{k_2}, \ldots) \mapsto G\{1, \ldots k_1, n_1 + 1 \ldots n_1 + k_2 \ldots \}$$

In other words we map a multiset to the orbit of the set that has $k_1$ elements among the first $n_1$, $k_2$ among the next $n_2$ etc. Note that this is indeed an orbit of the action of $G$. This gives the obvious definition of the inverse

$$B_n/G \to B_M$$

$$[S] \mapsto (1^{\sharp \{1,\ldots,n_1\} \cap S}, 2^{\sharp \{n_1+1,\ldots,n_2\} \cap S}, \ldots)$$

Again note that this map does not depend on the representative set $S$ and is well defined in the quotient.

These maps are easily seen to be mutually inverse to each other. Further it is clear that the respect the order relation.

It thus follows that $B_M$ is indeed rank-symmetric, rank-unimodal and Sperner.

**Problem 4** We will follow the same strategy as in Problem 3 and we will identify the poset $N_n$ with $B_n/G$ for some group $G$.

In this case we let $G = C_n$ the cyclic group generated by the $n$-cycle $(1, 2, \ldots n)$. 

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We will define mutually inverse bijections as above. To do this denote a $(0, 1)$-necklace in a linear arrangement by $\epsilon_1 \epsilon_2 \ldots \epsilon_n$, where $\epsilon_i \in \{0, 1\}$.

Now we construct the set corresponding to the indicator functor $(\epsilon_i)$, i.e., the set $S_{\epsilon_i}$ such that $k \in S_{\epsilon_i} \iff \epsilon_k = 1$. We then define the map in one direction by

$$N_n \rightarrow \mathcal{B}_n/G$$

$$(\epsilon_i) \mapsto [S_{\epsilon_i}]$$

Note that the $\epsilon_i$ are only defined up to cyclic permutation, but we are just consider the orbit of $S_{\epsilon_i}$ under the cyclic group, so this is well defined.

Similarly we can define the indicator function $\epsilon^S$ for a set $S$, such that $k \in S \iff \epsilon^S_k = 1$. We then can define the map.

$$\mathcal{B}_n/G \rightarrow N_n$$

$$[S] \mapsto (\epsilon^S_{\epsilon_i})$$

Again this is well-defined as both are defined up to cyclic permutation.

These two maps are obviously inverse to each other as sets and indicator functions are in bijection. They also obviously preserve the order again as the bijection between indicator functors and sets preserves the order.

It now follows just as in Problem 3 that this is a rank-symmetric, rank-unimodal and Sperner poset.

**Problem 5** Consider partitions of 7 and consider the elements in between $(4, 2, 1)$ and $(3, 2, 1)$. We then get the following poset

![Diagram](attachment:image.png)

These are indeed all the elements in between $(4, 2, 1)$ and $(3, 2, 1)$, since these elements have to have largest part being either 4 or 3 and the only other
partitions with those first parts are \((4, 3)\) and \((3, 1^4)\). There are also no other relations except those drawn in the diagram as can be easily checked. It follows that a maximal chain containing \((4, 2, 1) (4, 1^3)\) and \((3, 2, 1)\) is shorter than a chain containing \((4, 2, 1) (3, 2^2)\) and \((3, 2^2)\) and thus this cannot be graded. Further for any size larger than 7 we can pad the above partitions with 1’s to get that no poset of partitions of \(n\) for \(n \geq 7\) is graded. For partitions of size \(n < 7\) it can be directly checked that they are indeed graded.

For the second part note that \(\lambda_1' + \ldots + \lambda_k' = \sum_{r=1}^{k} \# \{ \lambda_i \geq r \} \). Further note that the statment \(\mu \geq \lambda \Leftrightarrow \lambda' \geq \mu'\) is symmetric, so we only need to prove one direction.

We prove \(\Leftarrow\). We prove the inequality inductively in \(k\). Assume we know it for \(k - 1\). If \(\mu_k \geq \lambda_k\) then the result follows from the \(k - 1\) inequality. So assume \(\mu_k \leq \lambda_k\). Then consider

\[
\sum_{i=1}^{k} (\lambda_i - \mu_k) \leq n - \sum_{r=1}^{\mu_k} \# \{ \lambda_i \geq r \} \leq n - \sum_{r=1}^{\mu_k} \# \{ \mu_i \geq r \} = \sum_{i=1}^{k} (\mu_i - \mu_k)
\]

The result now follows.

**Problem 6** We prove this by induction.

The case of \(n = 1\) is obvious.

Now if we know the case of \(n - 1\), we consider the \(S_{n-1}\) cosets on \(S_n\). The orbits are determined by \(\pi(k) = n\).

We now give a bijection between each orbit and \(S_{n-1}\). Denote by \(S_{n-1}^k \subset S_n\) the permutations satisfying \(\pi(k) = n\). For a permutation \(\sigma \in S_{n-1}\), we construct \(\pi^k_\sigma \in S_{n-1}^k\) by the formula

\[
\pi^k_\sigma(i) = \begin{cases} 
\sigma(i), & \text{if } i < k \\
n, & \text{if } i = k \\
\sigma(i-1), & \text{if } i > k
\end{cases}
\]
Note that $\text{inv}(\pi_k^\sigma) = \text{inv}(\sigma) + n - k$. Thus using induction we get

\[
\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = \sum_{k=1}^{n} \sum_{\sigma \in S_{n-1}} q^{\text{inv}(\pi_k^\sigma)}
\]

\[
= \sum_{k=1}^{n} \sum_{\sigma \in S_{n-1}} q^{\text{inv}(\sigma) + n - k}
\]

\[
= \sum_{k=1}^{n} q^{n-k} \sum_{\sigma \in S_{n-1}} q^{\text{inv}(\sigma)}
\]

\[
= [n-1]q! \sum_{k=1}^{n} q^{n-k}
\]

\[
= [n-1]q! [n]_q = [n]_q!
\]

There is also a geometric interpretation of this result. Let $G = \text{Gl}_n(F_q)$, $B$ the subgroup of upper triangular matrices. Then we have $G/B$ is the flag variety and has $[n]_q!$ points over $F_q$. But we can also use the Bruhat decomposition to get

\[
G/B = \Pi_{\sigma \in S_n} A^{\text{inv}(\sigma)}
\]

Thus the number of points can also be computed as $\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)}$.

**Problem 7** Consider $S = \{a_k < \cdots < a_1 < 0 = c < b_1 < \cdots < b_j\}$ here we only have $c$ if $S$ contains 0. For a subset $T \subset S$, we will associate $T^* \in M(j) \times M(k)^{op}$, where the map is one to one if $S$ does not contain 0 and two to one if $S$ contains 0. We do this as follows: Let $T = \{a_i, \ldots, a_i, b_{j_1}, \ldots, b_{j_l}\}$, then we associate $T^* = (\{j_1, \ldots, j_l\}, \{i_1, \ldots, i_r\})$.

Assume we have $\sum_{t \in T} t = \sum_{r \in R} r$ and $T, R \subset S$ distinct, then we will prove that $T^* = (\{j_1, \ldots, j_l\}, \{i_1, \ldots, i_r\})$ and $R^* = (\{j'_1, \ldots, j'_l\}, \{i'_1, \ldots, i'_r\})$ are incomparable in $M(j) \times M(k)^{op}$ unless they agree up to containing or not 0.

To see this assume for contradiction that $T^* \leq R^*$, then $a_i \leq a'_i$, because $i_n \geq i'_n$ and $a_k < \cdots < a_1$ and $T$ has maybe more negative elements. Similarly $b_{j_n} \leq b'_{j_n}$ just as above and with $R$ maybe containing more positive elements.

It thus follows $\sum_{t \in T} t \leq \sum_{r \in R} r$. Thus to have equality we must have both sets have the same negative and positive elements, so it can only differ on having or not $O$. 

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It thus follows using that that $M(j) \times M(k)^{op}$ is Sperner that we have
\[ f(S, \alpha) \leq \max \{ \#(M(j) \times M(k)^{op})_i \} \]
if $S$ does not contain 0 or twice the above if it does. Here $(M(j) \times M(k)^{op})_i$ is the $i$th rank elements, where a pair $(A, B)$ has rank $\sum_{a \in A} a - \sum_{b \in B} b$. Now we use that among all these sizes the maximal is $2(M(m) \times M(m)^{op})_0$ for all the sets $S$ of size $2m + 1$.
To see this note that the rank generating functions are
\[ \prod_{i=1}^{j}(1 + x^i) \prod_{s=1}^{k}(1 + x^{-s}) \]
Now using that these are rank-symmetric and rank-unimodular, we get that the middle size gets maximized by
\[ 2 \prod_{i=1}^{m}(1 + x^i) \prod_{s=1}^{m}(1 + x^{-s}) \]
Now note that this size is exactly given by $f(\{-m, \ldots, m\}, 0)$ and thus the result follows as required.

**Problem 8** Denote by $l(m, n)$ the number of partitions in $L(m, n)$ and $c(m, n)$ the number of coverings in $L(m, n)$.
We now find some recurrence relations.
First note that each partition in $L(m, n)$ either has the first part is of size $n$ or all the parts are of size $\leq n - 1$. Note that there are as many partitions with first part $n$ as there are partitions in $L(m - 1, n)$. Thus we get the recurrence realation
\[ l(m, n) = l(m - 1, n) + l(m, n - 1) \]
Thus we can prove this by induction on $m + n$. We prove by induction that $l(m, n) = \binom{m+n}{n}$. The initial cases are $l(m, 1) = m + 1$ $l(1, n) = n + 1$, which is easy to see. Thus by induction if we know it for the sum being $m + n - 1$. Thus we get
\[ l(m, n) = l(m-1,n)+l(m,n-1) = \binom{m+n-1}{n} + \binom{m+n-1}{n-1} = \binom{m+n}{n} \]
Thus the result follows by induction.

Now we find a recurrence between the number of coverings. Note that we can have coverings between two partitions that both have the first partition of size $n$, a covering between two partition who have no size $n$ part, or a covering between a partition with no parts of size $n$ and one with first part of size $n$. Note that obviously the first corresponds to the number of coverings in $L(m, n-1)$. The second corresponds to the number of covering in $L(m-1, n)$ and the last corresponds to the number of partitions with first part exactly $n-1$. Note that there are exactly as many partitions as in $L(m-1, n-1)$. Thus we get the recurrence

$$c(m, n) = c(m-1, n) + c(m, n-1) + l(m-1, n-1)$$

We again can prove the result about $c(m, n)$ using induction on $m+n$. Now we prove the desired result by induction. The initial cases are $c(m, 1) = m$ and $c(1, n) = n$ for which the result is clear. So assume we know the result for the sum being $m+n-1$ and then we get

$$c(m, n) = c(m-1, n) + c(m, n-1) + l(m-1, n-1)$$

$$= \frac{(m+n-2)!}{(m-2)!(n-1)!} + \frac{(m+n-2)!}{(m-1)!(n-2)!} + \frac{(m+n-2)!}{(m-1)!(n-1)!}$$

$$= \frac{(m+n-2)!}{(m-1)!(n-1)!}(m-1 + n - 1 + 1) = \frac{(m+n-1)!}{(m-1)!(n-1)!}$$