1. Give an example of a region $R$ in $\mathbb{Z}^2$ (that is, a finite collection of lattice unit squares) that has two domino tilings that cannot be connected by a sequence of flips in $2 \times 2$ squares.

2. Let $G$ be a planar bipartite graph. Each face of $G$ has an even number of sides. A Kasteleyn weighting of $G$ is a choice of sign for each undirected edge with the following property: if a face $F$ has $a$ sides, then $F$ has an odd number of $-$ signs if $a \equiv 0 \mod 4$ and an even number of $-$ signs if $a \equiv 2 \mod 4$. (Remark: our weighting from class is not a Kasteleyn weighting in this sense since we used imaginary numbers and not just signs.)

Let $K(G)$ be the weighted adjacency matrix of $G$ with a Kasteleyn weighting. Repeat the proof from class to show that

$$\# \{\text{perfect matchings of } G\} = \sqrt{|\det(K(G))|}.$$ 

3. Show that any planar bipartite graph $G$ has a Kasteleyn weighting as follows. Choose a spanning tree $T$ for $G$: that is, $T$ is a subtree of $G$ that uses all vertices of $G$. Now place the sign $+1$ on all the edges of $T$. Show that there is a unique way to place signs on the remaining edges of $G$ to obtain a Kasteleyn weighting.

4. Prove the following generalization of Lubell’s inequality. Suppose $A \subset B_n$ is a collection of subsets that does not contain any chain with $r+1$ elements. Then

$$\sum_{x \in A} \frac{1}{\binom{n}{|x|}} \leq r.$$ 

Use this to show that $|A|$ is bounded by the sum of the $r$ largest binomial coefficients $\binom{n}{k}$ with $k \in [0, n]$.

5. Let $P$ be a rank-symmetric, rank-unimodal poset. Show that if $P$ has a symmetric chain decomposition, then for any $j \geq 1$ the largest size of a union of $j$ antichains is equal to the largest size of a union of $j$ levels of $P$.

6. In this problem $\mathbb{F}_q$ denotes the finite field with $q = p^r$ elements, where $p$ is a prime and $r \geq 1$. Let $B_n(q)$ denote the poset of subspaces of $\mathbb{F}_q^n$, ordered by inclusion. The poset $B_n(q)$ is graded with rank function equal to the dimension.

   (1) Show that the number of $k$-dimensional subspaces of $\mathbb{F}_q^n$ is equal to

   $$\frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^n-k+1) - 1}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$ 

   (2) Show that every element $x \in B_n(q)$ with $\rho(x) = \dim(x) = k$ covers $[k]_q := 1 + q + q^2 + \cdots + q^{k-1}$ elements and is covered by $[n-k]_q := 1 + q + q^2 + \cdots + q^{n-k-1}$ elements.

   (3) Define up and down operators on $\mathbb{R}B_n(q)$ in the obvious way as we did for $B_n$. Show that they satisfy

   $$D_{i+1}U_i - U_{i-1}D_i = ([n-i]_q - [i]_q)I$$

   where $I$ is the identity transformation on $\mathbb{R}(B_n(q))_i$.

   (4) Conclude that $B_n(q)$ is Sperner.