Problem Set 2 Solutions

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**Problem 1 a)** Consider the graph $P_3$ of a path with 3 vertices, using the notation of Problem 3. Let the vertices be $v_1$, $v_2$ and $v_3$ using the same notation.

Now consider $H(v_1, v_2)$ and $H(v_2, v_1)$.

Clearly $H(v_1, v_2) = 1$ as the unique edge out of $v_1$ goes to $v_2$.

Now for $H(v_2, v_1)$ we have the recurrence

$$H(v_2, v_1) = \frac{1}{2} + \frac{1}{2}(2 + H(v_2, v_1))$$

Thus we get $H(v_2, v_1) = 3 \neq H(v_1, v_2)$

**b)** Now consider $H(v_1, v_2)$ after adding an extra edge joining $v_1$ and $v_3$.

Note that for the resulting triangle $H(v_1, v_2) = H(v_3, v_2)$, thus we get the recurrence

$$H(v_1, v_2) = \frac{1}{2} + \frac{1}{2}(1 + H(v_1, v_2))$$

Thus we get $H(v_1, v_2) = 2$ and so it increases after adding an edge.

**Problem 2** Note that $K_n$ is symmetric under any permutation, so $H(v_i, v_j)$ is independent of $i \neq j$. Thus we get the recurrence

$$H(v_i, v_j) = \frac{1}{n-1} + \frac{n-2}{n-1}(1 + H(v_i, v_j))$$

Thus we get $H(v_i, v_j) = n - 1$.
Problem 3 For the stationary state we need to find the eigenvector of eigenvalue 1 for the probability matrix

\[
M(G) = \begin{bmatrix}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 \\
0 & 1 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

This is given by the vector \(\frac{1}{2n-2} \begin{bmatrix} 1 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix}\).

For the second part of the question note that if \(i < j < k\) then to get from \(v_i\) to \(v_k\) you have to go through \(v_j\), thus we have

\[H(v_i, v_k) = H(v_i, v_j) + H(v_j, v_k)\]

Thus we can reduce this to understanding \(H(v_i, v_{i+1})\). Further we get the recursion

\[H(v_i, v_{i+1}) = \frac{1}{2} + \frac{1}{2}(1 + H(v_{i-1}, v_{i+1})) = 1 + \frac{1}{2}(H(v_{i-1}, v_i) + H(v_i, v_{i+1})\]

Thus we get

\[H(v_i, v_{i+1}) = 2 + H(v_{i-1}, v_i)\]

And note that \(H(v_1, v_2) = 1\), thus we get \(H(v_i, v_{i+1}) = 2i - 1\).

Then we get the

\[H(v_1, v_{i+1}) = \sum_{k=0}^{i} H(v_k, v_{k+1}) = \sum_{k=0}^{i} 2k - 1 = i^2\]

And so it follows that for \(i < j\) \(H(v_i, v_j) = H(v_1, v_j) - H(v_1, v_i) = (j - 1)^2 - (i - 1)^2\). By symmetry we can also consider the case \(i > j\) to get \(H(v_i, v_j) = (n - j)^2 - (n - i)^2\).
Problem 4 Note that $C_n$ is regular of degree $n$ and thus we have the probability matrix $M(G) = \frac{1}{n}A(G)$.
The questions asks to compute a diagonal entry of $(pI + (1-p)M(G))^l$. Note that all the diagonal entries are the same, so this is the same as computing $\frac{1}{n}tr((pI + (1-p)M(G))^l)$. To do this we can use the eigenvalues of this matrix. We know these from the previous Pset. In fact the eigenvalues of $A(G)$ are $\lambda = \epsilon_1 + \cdots + \epsilon_n \epsilon_i \in \{\pm 1\}$. Hence the eigenvalues of $pI + (1-p)M(G)$ are $p + (1-p)\frac{\lambda}{n}$. Now note that we can describe the eigenvalues of $A(G)$ as $n - 2k$ with multiplicity $\binom{n}{k}$. Thus we get the result is given by

$$\sum_{r=0}^{l} p^r(1-p)^{l-r} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{n-2k}{n}\right)^{l-r}$$

Problem 5 Note that the sum $H(u,v) + H(v,w) + H(w,u)$ computes the expected length of a closed walk starting at $u$ visiting $v$ and $w$ in that order and going back to $u$. We call these walks of type I.

Similarly $H(u,w) + H(w,v) + H(v,u)$ computed the expected length of a closed walk starting at $u$ visiting $w$ and $v$ in that order and going back to $u$. We call these walks of type II

Note that reversing the walk gives a bijection between walks of type I and walks of type II. Note that reversing the direction does not change the length, so to prove the equality we just need to prove that the probability to walk a particular path of Type I is the same as to walk the reverse path of Type II.

Assume the path is given by vertices $(v_0 = u, v_1, \ldots, v_n = u)$, then the probability to walk it forward is given by $\frac{1}{d_0} \frac{1}{d_1} \cdots \frac{1}{d_{n-1}}$, to walk in the reverse order the probability is $\frac{1}{d_0} \frac{1}{d_n} \cdots \frac{1}{d_{n-1}}$. But note that $v_0 = v_n = u$, thus the probabilities are indeed the same and the result follows.

Problem 6 Denote the relation we get as defined by $\preceq$. Then assume $u \preceq v$ and $v \preceq w$, ie $H(u,v) \leq H(v,u)$ and $H(v,w) \leq H(w,v)$, hence $H(u,v) + H(v,w) \leq H(u,w) + H(v,u)$.

Then we have $H(u,v) + H(v,w) + H(w,u) = H(u,w) + H(w,v) + H(v,u)$. Thus taking the difference we get $H(w,u) \geq H(u,w)$, thus we get $u \preceq w$, thus we have the relation is transitive.
**Problem 7** Let $T_v$ be the random variable that gives after how many step $v$ is visited for the first time and $R$ the random variable that measures the number of steps after which we visit half of the vertices. Note that the time when half are visited is the $n/2$th smallest $T_v$, ie there are at least $n/2 T_v$ larger than $R$. Thus we have $\sum_v T_v \geq \frac{n}{2} R$.

Now taking the expected value we get $\sum_v H(w, v) \geq \frac{n}{2} r$, thus we get $nH \geq \frac{n}{2} r$ and thus $2H \geq r$ as required.