18.212 Spring 2020 Problem Set 1
Due Wednesday February 19 midnight

Solutions must be typeset in LaTeX and either handed in class or emailed to me at tfylam@mit.edu.

1. Let $Z_n$, $n \geq 3$ denote the cycle graph with $n$ vertices and $n$ edges. Compute the eigenvalues of $Z_n$.

2. Let $K_{m,n}$ be the complete bipartite graph with parts of size $m$ and $n$. This is the simple graph with vertex set $V(K_{m,n}) = \{1,2,\ldots,m\} \cup \{1',2',\ldots,n'\}$ and an edge joining each $i$ to each $j'$. Compute the number of closed walks of length $\ell$ in $K_{m,n}$ and from this deduce the eigenvalues of $K_{m,n}$.

3. Let $G_1, G_2$ be two graphs. The direct product $G_1 \times G_2$ has vertex set $V(G_1) \times V(G_2)$, and for each vertex $(v_1, v_2)$ of $G_1 \times G_2$, and each edge $e \in E(G_1)$ connecting $v_1$ to $u_1$ (resp. each edge $e \in E(G_2)$ connecting $v_2$ to $u_2$), we have an edge $e' \in E(G_1 \times G_2)$ connecting $(v_1, v_2)$ to $(u_1, v_2)$ (resp. $(v_1, v_2)$ to $(v_1, u_2)$).
   - What are the eigenvalues of $G_1 \times G_2$ in terms of those of $G_1$ and $G_2$?
   - Use this to give another calculation of the eigenvalues of the cube $C_n$.

4. Let $G$ be a simple graph and $\bar{G}$ be the complement of $G$: thus $(u,v)$ is an edge in $\bar{G}$ if and only if it is not an edge in $G$. Determine the eigenvalues of $G$ in terms of those of $\bar{G}$.

5. Is it possible for $-1/3$ to be the eigenvalue of a graph?

6. Suppose that $G$ is a simple graph with $q$ edges. Show that $\lambda_1 \leq \sqrt{2q}$.

7. Suppose that $G$ is a nonempty simple graph such that for every two distinct vertices the number of common neighbors is odd. Show that $G$ has an odd number of vertices.