On the Geometry of the Strominger System

by

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Abstract

The Strominger system is a system of partial differential equations describing the
geometry of compactifications of heterotic superstrings with flux. Mathematically it
can be viewed as a generalization of Ricci-flat metrics on non-Kähler Calabi-Yau 3-
folds. In this thesis, I will present some explicit solutions to the Strominger system on
a class of noncompact Calabi-Yau 3-folds. These spaces include the important local
models like $\mathbb{C}^3$ as well as both deformed and resolved conifolds. Along the way, I also
give a new construction of non-Kähler Calabi-Yau 3-folds and prove a few results in
complex geometry.
Contents

1 Introduction 5

2 Preliminaries 9
  2.1 Basics on Complex Manifolds ........................................ 9
  2.2 Differential Geometry of Complex Vector Bundles ...................... 15
  2.3 SU(3) and G_2 Structures .............................................. 23
  2.4 Conifold Transition .................................................... 28
  2.5 Hyperkähler Manifolds and Their Twistor Spaces ...................... 30

3 The Geometry of the Strominger System 33
  3.1 Introduction .......................................................... 33
  3.2 An Example: Left-invariant Solutions on the Deformed Conifold ... 38
  3.3 Relation with G_2-structures ......................................... 43

4 A Class of Local Models 47
  4.1 The Geometry of Calabi-Gray Manifolds ............................... 47
  4.2 Degenerate Solutions on Calabi-Gray Manifolds ...................... 55
  4.3 Construction of Local Models ......................................... 61

A On Chern-Ricci-Flat Balanced Metrics 77

Bibliography 85

3
Chapter 1

Introduction

The marriage between mathematics and physics is one of the most exciting scientific developments in the second half of 20th century. Though many years have passed by, those sweet moments keep stirring up our minds, bringing unpredicted illuminations to our lives.

A particularly lovely story is the seminal contribution of Candelas-Horowitz-Strominger-Witten [25], where they embraced the remarkable world of Calabi-Yau geometries into string theory. To be precise, Candelas-Horowitz-Strominger-Witten discovered that, by considering 10d superstring theory on the metric product $M_4 \times X$, where $M_4$ is a maximally symmetric spacetime, $\mathcal{N} = 1$ spacetime supersymmetry effectively restricts the geometry of the internal manifold $X$. In particular, $X$ must be a complex 3-fold equipped with a holomorphic nowhere vanishing $(3,0)$-form $\Omega$ and a balanced (semi-Kähler) metric $\omega$. For the more familiar setting where the flux vanishes, $(X,\omega)$ has to be Kähler and Ricci-flat. Such geometric objects are more commonly known as Calabi-Yau spaces, thanks to the foundational work of Calabi [17, 18] and Yau [116, 118].

Replacing the metric product by a warped product, Strominger [103] derived a more general system of partial differential equations describing the geometry of compactification of heterotic superstrings with flux (torsion). This is the so-called Strominger system, the main subject to study in my thesis.

Among many other results, Strominger showed that in real dimension 6, the inter-
nal manifold $X$ has to be a complex 3-fold with trivial canonical bundle. Moreover, $X$ is equipped with a Hermitian metric $\omega$ and a Hermitian holomorphic vector bundle $(E, h)$. Let $\Omega$ be a nowhere vanishing holomorphic (3,0)-form on $X$. Then the Strominger system consists of the following equations:

\begin{align}
(1.1) & \quad d^* \omega = d^c \log \| \Omega \|_\omega, \\
(1.2) & \quad F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0, \\
(1.3) & \quad i\partial \bar{\partial} \omega = \frac{\alpha'}{4} (\text{Tr}(R \wedge R) - \text{Tr}(F \wedge F)).
\end{align}

In the above equations, $\alpha'$ is a positive coupling constant, while $R$ and $F$ are curvature 2-forms of $T^{1,0}X$ and $E$ respectively, computed with respect to certain metric connections. Equation (1.1) and (1.2) are consequences of $\mathcal{N} = 1$ supersymmetry, while Equations (1.3) comes from the Green-Schwarz anomaly cancellation mechanism.

Compared with its Calabi-Yau counterpart, the beauty and difficulty of the Strominger system lies in the fact that the inner manifold $X$ can be non-Kähler. Recall that a Hermitian manifold is a complex manifold equipped with a Hermitian metric, which can be characterized by a positive (1,1)-form $\omega$. The metric is called Kähler if $\omega$ is closed. We shall call a complex manifold non-Kähler if it does not support any Kähler metric.

Kähler manifolds have very beautiful properties, which arise from the compatibility of the complex-analytic and Riemannian structure. As a result we may employ both complex analytic and Riemannian techniques to study them. Such techniques have led to extremely elegant theories and theorems. To name a few, we have Hodge theory, Kodaira-Spencer’s deformation theory, Deligne-Griffiths-Morgan-Sullivan’s rational homotopy theory and so on.

Another great example in this line is Yau’s solution to the Calabi conjecture, as it stands at the intersection of nonlinear partial differential equations, complex algebraic geometry and theoretical physics. By solving a complex Monge-Ampère equation, Yau showed that within any fixed Kähler class on a compact Kähler manifold, there is a
unique Kähler metric with prescribed Ricci form. In particular, when the manifold has vanishing first Chern class, there exists a unique Ricci-flat Kähler metric in each Kähler class. Hence these Ricci-flat metrics can be regarded as canonical metrics in this Calabi-Yau setting.

However, when turning to the much broader kingdom of non-Kähler manifolds, we find ourselves disarmed. The failure of Kähler identities makes the Hodge theory not so satisfactory; the lack of Kähler form and \( \partial \bar{\partial} \)-lemma increases the complexity of Monge-Ampère type equation drastically. To summarize, we are short of tools to understand the non-Kähler world.

This situation may well be demonstrated in the problem of finding canonical Hermitian metrics. Nevertheless, there are still many things we can do. We shall approach canonical metrics on non-Kähler Calabi-Yau 3-folds through the study of the Strominger system, which is a natural generalization of Ricci-flat Kähler metrics from the viewpoint of heterotic string theory by turning on fluxes.

Besides the interest from physics, there are also mathematical motivations to understand the geometry of the Strominger system. The famous Reid’s fantasy [102] indicates that all the reasonably nice compact 3-folds with trivial canonical bundle can be connected with each other via conifold transitions, meanwhile the price to pay is to embrace the wild world of non-Kähler Calabi-Yau’s. Reid’s fantasy is very important in the study of moduli spaces of Calabi-Yau 3-folds, where a key problem is to understand the degeneration behavior on the boundary of moduli spaces. Therefore it would be very helpful if we can put good metrics on these Calabi-Yau 3-folds. For the Kähler ones, we have the canonical choice of Ricci-flat metrics; on the other hand, the Strominger system may serve as a guidance to “canonical” metrics on non-Kähler Calabi-Yau 3-folds.

Compared to the well-understood Kähler case, one of the biggest problems in understanding the Strominger system is the lack of nontrivial examples. In fact, it is not until more than twenty years later since Strominger’s work that the first non-perturbative solution was constructed by Fu and Yau [59]. In this thesis, I will provide some new explicit non-perturbative solutions to the Strominger system on a
class of noncompact Calabi-Yau 3-folds constructed from twistor spaces of hyperkähler 4-manifolds. The upshot is the following theorem.

**Theorem A.**
Let $N$ be a hyperkähler 4-manifold and let $p : Z \to \mathbb{CP}^1$ be its holomorphic twistor fibration. By removing an arbitrary fiber of $p$ from $Z$, we get a noncompact 3-fold $X$ which has trivial canonical bundle. For such $X$’s, we can always construct explicit solutions to the Strominger system on them.

In particular, the spaces described above contain $\mathbb{C}^3$ and the resolved conifold $\mathcal{O}(-1,-1)$ as special examples. These spaces are important local models for non-Kähler Calabi-Yau 3-folds. Therefore potentially we may use the solutions obtained in Theorem A to construct more general geometric models for compactification of heterotic superstrings.

This thesis is organized as follows. In Chapter 2 we review the necessary mathematical backgrounds for later use. Chapter 3 is an introduction to the geometry of Strominger system. As an example, we write down homogeneous solutions to the Strominger system on the deformed conifold $\text{SL}(2, \mathbb{C})$. Chapter 4 is devoted to the proof of Theorem A. Along the way we also provide a few related constructions and theorems in complex geometry.

It should be mentioned that some of the results presented in this thesis have already appeared in my joint work with my advisor S.-T. Yau [40] and my preprints [37, 38, 39].
Chapter 2

Preliminaries

2.1 Basics on Complex Manifolds

The goal of this section to review the basics on the theory of complex manifolds. All the materials can be found in the standard reference [81] if not cited otherwise.

**Definition 2.1.1.** Let $X$ be a smooth manifold of real dimension $n$. An *almost complex structure* on $M$ is a bundle isomorphism $J : TX \to TX$ such that $J^2 = -\text{id}$. If such a $J$ exists, then $n = 2m$ is even and $X$ is automatically oriented. In the language of $G$-structures, a choice of an almost complex structure $J$ is the same as a choice of a reduction of structure group from $\text{GL}(2m, \mathbb{R})$ to $\text{GL}(m, \mathbb{C})$.

**Definition 2.1.2.** We say $X$ is a *complex manifold* of complex dimension $m$ if $M$ as a topological space can be covered by coordinate charts homeomorphic to $\mathbb{C}^m$ such that the transition functions are holomorphic. A choice of the equivalence class of such coordinate charts is known as a *complex structure*.

A complex structure is automatically an almost complex structure in the following sense. Let \( \{ z^j = x^j + iy^j \}_{j=1}^m \) be a holomorphic coordinate chart of $X$, then we can define $J : TX \to TX$ by

\[
J \frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j} \quad \text{and} \quad J \frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}, \quad j = 1, \ldots, m.
\]
It is easy to see that this definition is independent of the choice of coordinate charts.

Let \((X, J)\) be an almost complex manifold. Since \(J\) is a real bundle map such that \(J^2 = -\text{id}\), we know that

\[
TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X,
\]

where \(T^{1,0}X\) and \(T^{0,1}X\) are the \(i\) and \(-i\) eigen-subbundles of \(TX \otimes \mathbb{C}\) with respect to \(J\). We say \(J\) is an integrable if \(T^{1,0}X\), as a complex distribution, is involutive. A famous theorem of Newlander-Nirenberg says that \(J\) comes from a complex manifold if and only if it is integrable, which is also equivalent to the vanishing of the Nijenhuis tensor

\[
\]

for any vector fields \(V, W\).

For an almost complex manifold \((X, J)\), we may treat \(J\) as an endomorphism of the cotangent bundle by defining \(J\alpha(V) := \alpha(JV)\) for any 1-form \(\alpha\) and vector field \(V\). Similarly we have the splitting of the complexified cotangent bundle

\[
T^*X \otimes \mathbb{C} = (T^*)^{1,0}X \oplus (T^*)^{0,1}X.
\]

In addition, we can define the bundle of \((p, q)\)-forms by

\[
\wedge^{p,q}T^*X := \wedge^p(T^*)^{1,0}X \otimes \wedge^q(T^*)^{0,1}X,
\]

and we have the decomposition of \(k\)-forms as sum of \((p, q)\)-forms

\[
\mathcal{A}^k(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X),
\]

where we use \(\mathcal{A}^*(X)\) to denote the space of smooth sections of \(\wedge^*T^*X\).

If \(J\) is integrable, then the exterior differential \(d\) restricted to \(\mathcal{A}^{p,q}(X)\) has at most two components:

\[
d\mathcal{A}^{p,q}(X) \subset \mathcal{A}^{p+1,q}(X) \oplus \mathcal{A}^{p,q+1}(X),
\]
hence we can define the first order differential operators $\partial$ and $\bar{\partial}$ by the corresponding projections of $d$. Clearly, we have

$$\partial^2 = \partial \bar{\partial} + \bar{\partial} \partial = \bar{\partial}^2 = 0.$$  

Sometimes it is useful to introduce the real operator

$$d^c := i(\bar{\partial} - \partial).$$

It follows that

$$dd^c = -d^c d = 2i\partial \bar{\partial}.$$  

As $\bar{\partial}^2 = 0$, $(\mathcal{A}^{p,\ast}(X), \bar{\partial})$ is a cochain complex and its associated cohomology groups are known as the Dolbeault cohomology groups

$$H^{p,q}(X) = \frac{\ker \left( \bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X) \right)}{\text{Im} \left( \bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X) \right)}.$$  

They can be identified with the sheaf cohomology associated to the holomorphic vector bundle $\Omega^p$ of $(p,0)$-forms

$$H^{p,q}(X) \cong H^q(X, \Omega^p).$$  

The dimensions of Dolbeault cohomology groups are known as the Hodge numbers

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X).$$

In most nice cases, for instance when $X$ is compact, these Hodge numbers are finite. Hodge numbers possess the symmetry $h^{p,q}(X) = h^{m-p,m-q}(X)$ coming from Serre duality. Moreover, the Frölicher [51] showed that there is a spectral sequence converging to the de Rham cohomology groups of $X$, whose $E_1$-page consists of exactly
the Dolbeault cohomology groups. As a corollary, we have

\[ b_k(X) \leq \sum_{p+q=k} h^{p,q}(X), \]

where \( b_k(X) \) is the \( k \)-th Betti number of \( X \).

Besides Dolbeault cohomology, there are many other kinds of cohomologies. Among others, we define the Bott-Chern cohomology \([14]\]

\[ H^{p,q}_{BC}(X) := \frac{\ker (d : A^{p,q}(X) \to A^{p+q+1}(X))}{\text{Im} \left( \partial \bar{\partial} : A^{p-1,q-1}(X) \to A^{p,q}(X) \right)}, \]

and the Aeppli cohomology \([2]\]

\[ H^{p,q}_A(X) := \frac{\ker \left( \partial \bar{\partial} : A^{p,q}(X) \to A^{p+1,q+1}(X) \right) - \text{Im} \left( \partial : A^{p-1,q}(X) \to A^{p,q}(X) \right) - \text{Im} \left( \bar{\partial} : A^{p,q-1}(X) \to A^{p,q}(X) \right)}{\text{Im} (\partial : A^{p-1,q}(X) \to A^{p,q}(X))}. \]

For compact complex manifolds, Bott-Chern and Aeppli cohomologies are finite dimensional. In general they are different from the Dolbeault cohomology.

**Definition 2.1.3.** Let \((X, J)\) be a complex manifold of complex dimension \(m\). A Hermitian metric on \(X\) is a Riemannian metric \(g\) compatible with \(J\) in the sense that \(g(JV, JW) = g(V, W)\) for any vector fields \(V\) and \(W\). A Hermitian metric is fully characterized by its associated positive \((1,1)\)-form defined by

\[ \omega(V, W) := g(JV, W). \]

A Hermitian metric \(\omega\) is called Kähler if \(d\omega = 0\).

Given \((X, J)\), Hermitian metrics always exist, and such a choice of Hermitian metric is equivalent to the choice of a reduction of structure group from \(GL(m, \mathbb{C})\) to \(U(m) = GL(m, \mathbb{C}) \cap SO(2m, \mathbb{R})\). However, the Levi-Civita connection associated to the Riemannian metric \(g\) does not necessarily descend to a connection on the principal \(U(m)\)-bundle. In fact, it descends if and only if \(g\) is a Kähler metric, or in other words, the holonomy group of \((X, g)\) is a subgroup of \(U(m)\).
Compact Kähler manifolds behave well in terms of Hodge theory. It is a well-known fact that for a compact Kähler manifold $X$, the Frölicher spectral sequence degenerates at $E_1$-page and we have the Hodge decomposition

$$H^k(X; \mathbb{C}) = \sum_{p+q=k} H^{p,q}(X).$$

Consequently we see the extra Hodge symmetry $h^{p,q}(X) = h^{q,p}(X)$ and the equality

$$b_k(X) = \sum_{p+q=k} h^{p,q}(X).$$

As a corollary, the odd Betti numbers of $X$ are even. Moreover, $X$ satisfy the so-called $\partial \bar{\partial}$-lemma. One version of the $\partial \bar{\partial}$-lemma dictates that if a $(p, q)$-form $\alpha$ is both $\partial$-closed and $\bar{\partial}$-exact, then it must be $\partial \bar{\partial}$-exact. It follows entirely from the $\partial \bar{\partial}$-lemma that the Bott-Chern cohomology and Aeppli cohomology coincide with the Dolbeault cohomology. In fact, the $\partial \bar{\partial}$-lemma is slightly stronger than the degeneracy of Frölicher spectral sequence. It was proved by Deligne-Griffiths-Morgan-Sullivan [33] that the $\partial \bar{\partial}$-lemma is equivalent to the degeneracy of Frölicher spectral sequence at $E_1$-page plus a Hodge structure condition.

The $\partial \bar{\partial}$-lemma holds for a strictly larger class of compact complex manifolds than the Kählerian ones. Recall that a compact complex manifold is said to be of Fujiki class $\mathcal{C}$ if it is bimeromorphic to a compact Kähler manifold. It was proved by Deligne-Griffiths-Morgan-Sullivan [33] that manifolds of Fujiki class $\mathcal{C}$ always satisfy the $\partial \bar{\partial}$-lemma. It is also noteworthy to point out that though the Kähler condition [84] and the $\partial \bar{\partial}$-lemma [113, 115] are stable under small deformations, the Fujiki class $\mathcal{C}$ is not stable under small deformations [23, 90].

Besides the restrictions on odd Betti numbers, there are many topological and geometric obstructions to the existence of Kähler metrics on a compact complex manifold. For example, the fundamental group of a compact Kähler manifold has to be a so-called “Kähler group”; any nontrivial complex submanifold of a compact Kähler manifold cannot be homologous to 0. Furthermore, we have the following
intrinsic characterization of compact Kähler manifolds in terms of geometric measure theory:

**Theorem 2.1.4** (Harvey-Lawson [72]).

Suppose $X$ is a compact complex manifold, then $X$ admits a Kähler metric if and only if there are no positive currents on $X$ which are the $(1, 1)$-component of boundaries.

In order to understand the much broader world of non-Kähler manifolds, it is natural to consider Hermitian metrics with weaker-than-Kähler conditions. In this thesis, we will only deal with balanced (semi-Kähler), Gauduchon, pluriclosed (strong Kähler with torsion), and astheno-Kähler metrics.

**Definition 2.1.5.** Following Michelsohn [96], we say a Hermitian metric $\omega$ on a complex $m$-fold $X$ is balanced (also known as semi-Kähler in old literatures) if

$$d(\omega^{m-1}) = 0.$$ 

In particular in complex dimension 2, balanced metrics are exactly Kähler metrics.

It is a simple exercise of linear algebra that $d(\omega^k) = 0$ for some $k < m - 1$ implies that $\omega$ is Kähler. The balanced condition can be interpreted as $d^* \omega = 0$, where $d^* = -*d*$ is the adjoint operator of $d$. Hence one should think of a balanced metric as some notion dual to a Kähler metric. Indeed this is the case as demonstrated in [96]. In particular, Michelsohn gave the following intrinsic characterization of balanced manifolds dual to Theorem 2.1.4:

**Theorem 2.1.6** (Michelsohn [96]).

Let $X$ be a compact complex manifold of complex dimension $m$. Then $X$ admits a balanced metric if and only if there are no positive currents on $X$ which are the $(m - 1, m - 1)$-component of boundaries.

There are many non-Kähler manifolds that are balanced. For example, Alessandrini-Bassanelli [4] showed that being balanced is preserved under modification, hence all the compact complex manifolds of Fujiki class $C$ are balanced.
Definition 2.1.7. Let $X$ be a complex manifold of complex dimension $m$. We say a Hermitian metric $\omega$ on $X$ is Gauduchon if $i\partial\bar{\partial}(\omega^{m-1}) = 0$.

Unlike for balanced metrics, there are no obstructions to the existence of Gauduchon metrics. In fact, we have

Theorem 2.1.8 (Gauduchon [61, 62]).
Let $X$ be a compact complex manifold with complex dimension at least 2. For any Hermitian metric on $X$, there exists a unique Gauduchon metric in its conformal class up to scaling.

Definition 2.1.9. A Hermitian metric $\omega$ on a complex $m$-fold is called pluriclosed (a.k.a. SKT, standing for strong Kähler with torsion), if $i\partial\bar{\partial}\omega = 0$. It is known as an astheno Kähler metric [82] if instead $i\partial\bar{\partial}(\omega^{m-2}) = 0$. Notice that for 3-folds, these two concepts coincide. It is also known that there are compact complex manifolds with no pluriclosed/astheno Kähler metrics.

Balanced, pluriclosed and astheno Kähler metrics have been extensively studied in the vast literature of non-Kähler geometry. We shall refer to the survey papers [52, 45, 46] and the references therein for more information about these metrics.

2.2 Differential Geometry of Complex Vector Bundles

In this section, we will review the theory of complex and holomorphic vector bundles. Most material are standard and can be found in [81]. The theory of Hermitian connections on tangent bundle is taken from [63].

Let $X$ be a smooth manifold and $E$ a smooth complex vector bundle over $X$. A connection $\nabla$ on $E$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ satisfying

$$\nabla(fs) = f\nabla s + df \otimes s$$

for any $f \in \mathcal{A}^0(X), s \in \mathcal{A}^0(E)$. 

15
where $\mathcal{A}^k(E)$ is the space of $E$-valued complex $k$-forms on $X$. By a partition of unity argument we know that connections always exist and they form an affine space modeled on $\mathcal{A}^1(\text{End } E)$.

The curvature form $F^\nabla$ associated to the connection $\nabla$ is defined to be

$$F^\nabla = \nabla^2 \in \mathcal{A}^2(\text{End } E).$$

The famous Chern-Weil theory says that the Chern classes can be represented by curvature forms. More precisely, we have

$$c(E) = 1 + c_1(E) + \cdots + c_m(E) = \det \left( I + \frac{i}{2\pi} F^\nabla \right) = 1 + \frac{i \cdot \text{Tr } F^\nabla}{2\pi} + \frac{\text{Tr}(F^\nabla)^2 - (\text{Tr } F^\nabla)^2}{8\pi^2} + \ldots$$

In the above equation, the Chern classes should be understood as de Rham cohomology classes, while the second line says that these cohomology classes can be represented by closed forms given by trace of powers of $F^\nabla$. In particular, $\text{Tr}(F^\nabla)^k$ are closed forms and their de Rham cohomology classes are independent of the choice of connections.

Now let $X$ be a complex manifold. We say $E$ is a holomorphic vector bundle over $X$ if we can find local trivializations of $E \to X$ covering $X$ such that the transition functions are holomorphic. Given a holomorphic vector bundle $E$ over $X$, we can define the $\bar{\partial}$-operator and get the cochain complex $\bar{\partial} : \mathcal{A}^{0,q}(E) \to \mathcal{A}^{0,q+1}(E)$. Like the differential form case, its cohomology computes the sheaf cohomology of the locally free sheaf associated to $E$.

Now let $E$ be a holomorphic vector bundle over $X$. When $E$ is equipped with a Hermitian metric $\langle \cdot, \cdot \rangle$, there is a canonical choice of connection $\nabla^c$, known as the Chern connection (it is called Hermitian connection in physics literature). The Chern
connection is uniquely characterized by the following properties

\[
\begin{cases}
\langle \nabla^c \rangle^{0,1} = \bar{\partial}, \\
\text{d} \langle s_1, s_2 \rangle = \langle \nabla^c s_1, s_2 \rangle + \langle s_1, \nabla^c s_2 \rangle, \text{ for any local sections } s_1, s_2 \text{ of } E.
\end{cases}
\]

Roughly speaking, the first condition says that $\nabla^c$ is compatible with the holomorphic structure while the second condition says that $\nabla^c$ is compatible with the Hermitian metric.

By choosing a local holomorphic frame $\{s_1, \ldots, s_r\}$ of $E$, we can express the Hermitian metric by the Hermitian matrix $H = (h_{jk})_{r \times r}$, where $h_{jk} = \langle s_j, s_k \rangle$. Then the curvature form $F^{\nabla^c}$ associated to the Chern connection is given by

\[
F^{\nabla^c} = \bar{\partial} \left( H^{-1} \partial H \right) \in \mathcal{A}^{1,1}(\text{End } E).
\]

As a consequence all the Chern forms $c^{\nabla^c}_k(E, h)$ are real $(k, k)$-forms and their Bott-Chern cohomology classes

\[
c^{BC}_k(E) \in H^{k,k}_{BC}(X; \mathbb{R})
\]

are independent of the choice of the Hermitian metric [14]. In particular, when $k = 1$, the first Chern form can be computed by

\[
c^1_{\nabla^c}(E) = -\frac{i}{2\pi} \partial \bar{\partial} \log \det H \in H^{1,1}_{BC}(X; \mathbb{R}).
\]

As an analogue of the Newlander-Nirenberg theorem, the holomorphic structure of $E$ can be recovered from a connection whose curvature form has vanishing $(0, 2)$-component, this is the famous Koszul-Malgrange integrability theorem [85].

Now let $X$ be a compact complex manifold of complex dimension $m$ with a Gauduchon metric $\omega$. Let $E$ be a holomorphic vector bundle over $X$. The degree of $E$ with respect to the polarization $\omega$ is defined to be

\[
\deg(E) := \int_X c^{BC}_1(E) \cdot \frac{\omega^{m-1}}{(m-1)!}.
\]
The Gauduchon condition guarantees that the above definition is well-defined in the sense that it does not depend on the representative of the Bott-Chern cohomology. In addition, the degree is topological if \( \omega \) is a balanced metric, in the sense that \( \deg(E) \) depends only on the de Rham cohomology class \([\omega^{m-1}]\) and the topology of \( E \). The *slope* of \( E \) is defined to be

\[
\mu(E) = \frac{\deg(E)}{\text{rank}(E)}.
\]

By taking resolutions, we can generalize the notion of slope to coherent analytic sheaves.

**Definition 2.2.1.** We say \( E \) is *slope-stable* (*slope-semistable*) if for any subsheaf \( \mathcal{F} \subset E \) with \( \text{rank}(\mathcal{F}) < \text{rank}(E) \), we have

\[
\mu(\mathcal{F}) < (\leq) \mu(E).
\]

We say \( E \) is *slope-polystable* if it is holomorphically a direct sum of stable subbundles with same slope.

**Definition 2.2.2.** Let \( E \) be a holomorphic vector bundle over \( X \). We say a Hermitian metric \( h \) on \( E \) is Hermitian-Yang-Mills (Hermitian-Einstein) if

\[
i\Lambda F \nabla c = \gamma_h \cdot \text{id}_E,
\]

or equivalently

\[
iF \nabla c \wedge \frac{\omega^{m-1}}{(m-1)!} = \gamma_h \cdot \text{id}_E \frac{\omega^m}{m!},
\]

where \( \gamma_h \) is a constant and \( \Lambda \) is the operator of contracting \( \omega \).

The celebrated Donaldson-Uhlenbeck-Yau Theorem says that the slope stability is equivalent to the solvability of Hermitian-Yang-Mills equation in the sense that

**Theorem 2.2.3** (Donaldson-Uhlenbeck-Yau [34, 111, 92]).

Let \((X, \omega)\) be a compact complex manifold with a Gauduchon metric. A holomorphic vector bundle \( E \) over \( X \) admits a solution to the Hermitian-Yang-Mills equation if and only if it is slope-polystable with respect to \( \omega \).
From now on in this section, we restrict ourselves from general holomorphic vector bundles to the holomorphic tangent bundle. As a complex vector bundle, the holomorphic tangent bundle $T^{1,0}X$ can be naturally identified with $(TX, J)$. Under such an identification, connections on $T^{1,0}X$ are exactly those real connections on $TX$ such that $J$ is parallel. Suppose now $X$ is equipped with a Hermitian metric $\omega$, then we have the associated Levi-Civita connection $\nabla^{LC}$ and the Chern connection $\nabla^{c}$. It is a well-known fact that these two connections coincide if and only if $\omega$ is Kähler. In fact, there are lots of “canonical” connections on a general Hermitian manifold.

Let $(X, J, g)$ be a Hermitian manifold of complex dimension $m$. We will use $(\cdot, \cdot)$ to denote the Hermitian inner product and $(\cdot, \cdot)$ the (complexified) Riemannian inner product. Following [63], we will study Hermitian connections on $X$, i.e. those real connections $D$ on $TX$ satisfying $Dg = 0$ and $DJ = 0$.

The first step is to understand the space of $TX$-valued real 2-forms. We will use $A^2(TX)$ to denote the space of $TX$-valued 2-forms on $X$.

Each element $B \in A^2(TX)$ will be also be identified (via $g$) tacitly as a trilinear form which is skew-symmetric with respect to the last two arguments, by

$$B(U, V, W) = g(U, B(V, W)).$$

In particular, the space of 3-forms $A^3(X)$ will be considered as a subspace of $A^2(TX)$. Let $b : A^2(TX) \to A^3(X)$ be the Bianchi projection operator given by

$$(bB)(U, V, W) = \frac{1}{3}(B(U, V, W) + B(V, W, U) + B(W, U, V)).$$

The trace of $B$ is the 1-form $\text{Tr}(B)$ defined by contracting the first two arguments, i.e.

$$\text{Tr}(B)(W) = \sum_i B(e_i, e_i, W),$$

where $\{e_i\}$ is an orthonormal frame of $X$ with respect to $g$. The trace should be thought as a projection operator from $A^2(TX)$ onto $A^1(X)$, where the latter is real-
ized as a subspace of $\mathcal{A}^2(TX)$ by identifying $\alpha \in \mathcal{A}^1(X)$ with $\tilde{\alpha} \in \mathcal{A}^2(TX)$ via

$$\tilde{\alpha}(U, V, W) = \frac{1}{2m-1}(\alpha(W)g(U, V) - \alpha(V)g(W, U)).$$

It is straightforward to check that

$$\text{Tr}(\tilde{\alpha}) = \alpha.$$

We thus get a decomposition of $\mathcal{A}^2(TX)$:

$$\mathcal{A}^2(TX) = \mathcal{A}^1(X) \oplus \mathcal{A}^3(X) \oplus (\mathcal{A}^2(TX))^0,$$

where $(\mathcal{A}^2(TX))^0$ is the subspace of traceless elements satisfying the Bianchi identity. Accordingly, we can express $B \in \mathcal{A}^2(TX)$ as

$$B = \text{Tr}(B) + bB + B^0.$$

Up to now, everything we did works for general Riemannian manifolds. Now we shall take $J$ into account.

**Definition 2.2.4.** An element $B \in \mathcal{A}^2(TX)$ is said to be of

(a). type $(1,1)$, if $B(JV, JW) = B(V, W),$

(b). type $(2,0)$, if $B(JV, W) = JB(V, W),$

(c). type $(0,2)$, if $B(JV, W) = -JB(V, W).$

We shall denote the corresponding spaces $\mathcal{A}^{1,1}(TX), \mathcal{A}^{2,0}(TX)$ and $\mathcal{A}^{0,2}(TX)$ respectively.

We also introduce an involution $\mathfrak{m}$ on $\mathcal{A}^2(TX)$ defined by

$$\mathfrak{m}B(U, V, W) = B(U, JV, JW).$$
Let $\mathcal{M}_{\pm 1}$ be the eigenspaces of $\mathcal{M}$ with eigenvalues $\pm 1$. It is clear that

$$\mathcal{M}_1 = \mathcal{A}^{1,1}(TX).$$

We can further introduce an involution $\mathfrak{N}$ on $\mathcal{M}_{-1}$ by

$$\mathfrak{N}B(V, W) = JB(JV, W).$$

Again, we have

$$\mathfrak{N}_1 = \mathcal{A}^{0,2}(TX) \quad \text{and} \quad \mathfrak{N}_{-1} = \mathcal{A}^{2,0}(TX).$$

Hence we conclude that

$$\mathcal{A}^2(TX) = \mathcal{A}^{1,1}(TX) \oplus \mathcal{A}^{2,0}(TX) \oplus \mathcal{A}^{0,2}(TX).$$

Fix the Chern connection $\nabla^c$ on $(X, J, g)$. For any $A \in \mathcal{A}^2(TX)$, we can define a connection $D^A$ by letting

$$g(D^A_U V, W) - g(\nabla^c_U V, W) = A(U, V, W).$$

We shall call $A$ the potential of $D^A$.

It is clear that $D^A$ always preserves $g$ and $D^A J = 0$ if and only if $A \in \mathcal{A}^{1,1}(TX)$. Therefore the space of Hermitian connections is an affine space modeled on $\mathcal{A}^{1,1}(TX)$.

In particular, for any real 3-form $B \in \mathcal{A}^3(X)$, we can use it to twist the Chern connection to get a Hermitian connection $D^B$ with potential $B + \mathfrak{M}B$.

It is easy to check that the $(3, 0) + (0, 3)$-part of $B$ does not contribute to $B + \mathfrak{M}B$, therefore without loss of generality, we may assume that in local coordinates

$$B = B_{jkld} dz^j \wedge dz^k \wedge d\bar{z}^l + B_{ljk} dz^l \wedge d\bar{z}^j \wedge d\bar{z}^k,$$

where we have

$$\overline{B_{jkld}} = B_{ljk}.$$
It is straightforward to calculate that

\[ B + \mathfrak{M}B = 4B_{jk\ell}dz^j \otimes dz^k \wedge dz^\ell + (\text{conjugate}). \]

In order to compute the curvature forms associated to \( D^B \), we need to identify \( B + \mathfrak{M}B \) as an element in \( \mathcal{A}^1(\text{End}(T^{1,0}X)) \). Let \( \hat{B} \) denote this element. A detailed calculation shows that with respect to the frame

\[ \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m} \right\}^T, \]

the potential \( \hat{B} \) can be expressed in the matrix form

\[ \hat{B}^i_s = 4(B_{jsk}dz^j - B_{sjk}dz^j)h^{ks}. \]

In particular we have

\[ \text{Tr}(\hat{B}) = 4(B_{jsk}dz^j - B_{sjk}dz^j)h^{ks} = 2i\Lambda B. \]

Hence we have proved

**Proposition 2.2.5.**

\[ c_1^{DB}(X) = \frac{i}{2\pi} \text{Tr}(F^{DB}) = \frac{i}{2\pi} \left( \text{Tr}(F^\nabla) + d\text{Tr}(\hat{B}) \right) = c_1^{\nabla}(X) - d \left( \frac{\Lambda B}{\pi} \right). \]

The space of \( B \)-twisted Hermitian connections is still too big for us. To get a much smaller space, we may make a canonical choice of \( B \). By setting \( B \propto d^c\omega \), we get the so-called canonical 1-parameter family of Hermitian connections.

The canonical 1-parameter family of Hermitian connections \( \nabla^t \) is defined by

\[ \nabla^t = \nabla^c + \frac{t-1}{4}(d^c\omega + \mathfrak{M}(d^c\omega)), \]

where we have to identify the 3-form \( d^c\omega \) as an element of \( \mathcal{A}^2(TX) \). This affine line parameterizes all the known “canonical” Hermitian connections.
(a). \( t = 0 \), it is known as the first canonical connection of Lichnerowicz.

(b). \( t = 1 \), it is the Chern connection \( \nabla^c \).

(c). \( t = -1 \), this is the Bismut-Strominger connection \( \nabla^b \). It is the unique Hermitian connection such that its torsion tensor \( T^b \) is totally skew-symmetric. In particular, the torsion tensor \( T^b = -d^c \omega \) can be related to the flux term \( H \) in string theory. Moreover, \( \nabla^b \) and its analogue in \( G_2 \)-geometry are widely used in mathematical physics.

(d). \( t = 1/2 \), it has been called the conformal connection by Libermann.

(e). \( t = 1/3 \), this is the Hermitian connection that minimizes the norm of its torsion tensor.

When \( X \) is Kähler, this line collapses to a single point, i.e. the Levi-Civita connection. As a corollary of Proposition 2.2.5, we know that

\[
(2.1) \quad c_1^{\nabla^b}(X) = c_1^{\nabla^c}(X) + \frac{1}{2\pi} d(\Lambda d^c \omega).
\]

### 2.3 SU(3) and \( G_2 \) Structures

Let \( M \) be an oriented Riemannian \( m \)-manifold and let \( G \) be a connected closed Lie subgroup of \( SO(m) \). A \( G \)-structure on \( M \) is a reduction of the frame bundle of \( M \) to a principal \( G \)-subbundle. The holonomy group of \( M \) is contained in \( G \) if and only if the Levi-Civita connection reduces to a \( G \)-connection simultaneously. The obstruction for the reduction of Levi-Civita connection is given by the intrinsic torsion, which pointwise is an element of \( T^*M \otimes g^\perp \), where \( g \) is the Lie algebra of \( G \) identified as a subspace of 2-forms on \( M \), and \( \perp \) denotes the orthogonal complement.

According to Berger’s classification list, the only possible holonomy groups for an irreducible non-symmetric Riemannian manifold are the series \( SO(n) \), \( U(n) \), \( SU(n) \), \( Sp(n) \), \( Sp(n) \cdot Sp(1) \) and the exceptional ones \( G_2 \) and \( Spin(7) \). Manifolds of special holonomy play an important role in the string theory, especially for \( SU(n) \)-manifolds.
(Calabi-Yau) and $G_2$-manifolds. The relation $SU(2) \subset SU(3) \subset G_2$ is closely related to various string dualities. Mathematically this relation is used to construct various compact $G_2$-manifolds [83, 86]. In this section, we will first review the consequences of this relation in the setting of $G$-structures with torsion. Then we will explain how the classical constructions of Calabi [19] and Gray [67] can be interpreted in our language.

Let $V$ be a finite dimensional real vector space. Recall from [76] that a $p$-form $\varphi \in \wedge^p V^*$ is called *stable* if its orbit under the natural $GL(V)$-action is an open subset of $\wedge^p V^*$.

It is classically known that stable forms occur only in the following cases:

- $p = 1$, arbitrary $n \in \mathbb{Z}_+$.
- $p = 2$, arbitrary $n \in \mathbb{Z}_+$.
- $p = 3$, $n = 6, 7$ or 8.
- The dual of each above situations. That is, if the space of $p$-forms on $V$ has an open orbit, so does the space of $(n - p)$-forms.

In this section, we will focus on the case $p = 3$ and $n = 6, 7$. A more detailed account of geometries associated to stable forms can be found in [75, 76, 77, 39].

For $p = 3$ and $n = 6$, there are two open $GL(V)$-open orbits in $\wedge^3 V^*$. The one we are interested in has stabilizer isomorphic to $SL(3, \mathbb{C})$, which we denote by $\mathcal{O}_6^-(V)$. For any $\Omega_1 \in \mathcal{O}_6^-(V)$, it naturally defines a complex structure $J$ on $V$ such that $\Omega_1$ is the real part of a nowhere-vanishing $(3, 0)$-form. With a suitable choice of basis $e^1, e^2, \ldots, e^6$ of $V^*$ such that $e^{k+3} = Je^k$ for $k = 1, 2, 3$, our $\Omega_1$ can be expressed as

$$\Omega_1 = e^1 \wedge e^2 \wedge e^3 - e^1 \wedge e^5 \wedge e^6 + e^2 \wedge e^4 \wedge e^6 - e^3 \wedge e^4 \wedge e^5$$

$$= \text{Re} (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6).$$

For $p = 3$ and $n = 7$, there are also two open $GL(V)$-orbits in $\wedge^3 V^*$. We are interested in one of them, denoted by $\mathcal{O}_7^-(V)$, whose stabilizer is isomorphic to the compact exceptional Lie group $G_2$. For each $\varphi \in \mathcal{O}_7^-(V)$, it naturally defines a
Riemannian metric on $V$. By a suitable choice of orthonormal basis $e^1, \ldots, e^7$ of $V^*$, we can express $\varphi$ as

$$\varphi = e^1 \wedge e^2 \wedge e^3 - e^1 \wedge e^6 \wedge e^7 + e^2 \wedge e^5 \wedge e^7 - e^3 \wedge e^5 \wedge e^6 + e^1 \wedge e^4 \wedge e^6 + e^2 \wedge e^4 \wedge e^7 + e^3 \wedge e^4 \wedge e^7.$$

Let $W$ be any 6-dimensional subspace of $V$, then $\varphi|_W$ lies in the orbit $O_6^-(W)$. Moreover, $\varphi|_W$ together with the induced metric on $W$ defines an SU(3)-structure on $W$.

Notice that for an oriented 7-manifold $\tilde{M}$, giving a 3-form $\varphi$ lying in the orbit $O_7^-(T_x\tilde{M})$ for every $x \in \tilde{M}$ is equivalent to giving a $G_2$-structure on $\tilde{M}$. Therefore we have

**Theorem 2.3.1** (Calabi [19], Gray [67]).
Let $\tilde{M}$ be a 7-manifold with a $G_2$-structure $\varphi$. For any immersed oriented hypersurface $M$ of $\tilde{M}$, there is a natural SU(3)-structure induced by $\varphi$.

Calabi-Gray’s construction produces lots of almost complex 6-manifolds including $S^6$. It is a natural question to ask when such almost complex structures are integrable. The necessary and sufficient condition for integrability was derived in [19, 67]. In particular, by making use of SU(2) $\subset G_2$, Calabi and Gray proved

**Theorem 2.3.2** (Calabi [19], Gray [67]).
Let $\tilde{M} = T^3 \times N$ for $N = T^4$ or the K3 surface, equipped with a $G_2$-metric. If $\Sigma_g \subset T^3$ is a minimal surface of genus $g$ in flat $T^3$, then the almost complex structure on $M = \Sigma_g \times N$ constructed above is integrable and $M$ is non-Kähler. Moreover, the projection $\pi : M \to \Sigma_g$ is holomorphic, and the naturally induced metric on $M$ is balanced.

According to Meeks [95] and Traizet [108], minimal surfaces in $T^3$ (classically known as triply periodic minimal surfaces in $\mathbb{R}^3$) exist for all $g \geq 3$. Using this construction, Calabi gave the first example showing that $c_1$ of a complex manifold is not a smooth invariant, thus answering a question asked by Hirzebruch. It was
noticed in [39] that such constructed $M$’s have trivial canonical bundle, which follows from a slightly more general proposition:

**Proposition 2.3.3.**

Let $\tilde{M}$ be a 7-manifold with a $G_2$-structure $\varphi$ such that $d\varphi = 0$. If $M \subset \tilde{M}$ is an immersed oriented hypersurface such the induced almost complex structure on $M$ is integrable, then $M$ has holomorphically trivial canonical bundle.

**Proof.** As $M$ has an SU(3)-structure, we can choose $\Omega = \Omega_1 + i\Omega_2$ to be a nowhere vanishing $(3,0)$-form on $M$. By the construction above, we may assume that $\Omega_1 = \varphi|_M$, therefore

$$d\Omega = d\Omega_1 + id\Omega_2 = id\Omega_2.$$ 

Since the almost complex structure is integrable, we know that $d\Omega$ is a $(3,1)$-form. Notice that $d\Omega_2$ is real, so the only possibility is that $d\Omega = 0$. \hfill $\square$

We will call the non-Kähler Calabi-Yau 3-folds in Theorem 2.3.2 the *Calabi-Gray manifolds*. Their complex geometry will be studied in Chapter 4 in detail.

Roughly speaking, allowing nonzero flux in the superstring theory is equivalent to allowing torsional $G$-structures on the space where strings are compactified. For this reason, we are interested in SU(3) and $G_2$ structures with torsion.

The idea of using representation theory to classify intrinsic torsions was first developed by Gray-Hervella [68], where they divided almost Hermitian geometries, i.e. $U(m)$-structures, into 16 classes according to their torsion (see also [36]). Similar story was also carried out for $G_2$-structures [41, 15]. The case of SU(3)-structures and their relations to $G_2$-structures can be found in [29]. Now let us review the theory of torsional SU(3) and $G_2$ structures.

Let us first consider a $U(3)$-structure on a 6-manifold $M$. The space $T^*M \otimes u(3)\perp$ decomposes as 4 irreducible $U(3)$-representations

$$T^*M \otimes u(3)\perp = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

of real dimension 2, 16, 12 and 6 respectively, where $V_4$ is isomorphic to the standard
representation of $U(3)$ on $\mathbb{C}^3 = \mathbb{R}^6$. It is well-known that both $V_1$ and $V_2$ components of intrinsic torsion vanishes if and only if the almost complex structure is integrable; while the $V_4$-component vanishes if and only if the metric is almost balanced, i.e. $d(\omega^2) = 0$.

When we turn to $SU(3)$-structures, notice that $\mathfrak{su}(3)^\perp = \mathfrak{u}(3)^\perp \oplus \mathbb{R}$, so

$$T^*M \otimes \mathfrak{su}(3)^\perp = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5,$$

where the extra component $V_5$ is also isomorphic to the standard representation of $SU(3)$ on $\mathbb{C}^3 = \mathbb{R}^6$.

For the $SU(3)$-structure appearing in the Strominger system, we know from above that both $V_1$ and $V_2$ components of intrinsic torsion vanish. Moreover, the conformally balanced equation (3.4) tells us that [26]

$$2V_4 + V_5 = 0 \text{ and both } V_4 \text{ and } V_5 \text{ are exact.}$$

If in addition the metric is balanced, both $V_4$ and $V_5$ components vanish.

For $G_2$-structures on a 7-manifold $\bar{M}$, their intrinsic torsions can also be decomposed into 4 irreducible components

$$T^*\bar{M} \otimes g_2^\perp = W_1 \oplus W_2 \oplus W_3 \oplus W_4$$

of real dimension 1, 14, 27 and 7 respectively.

The relevant class of $G_2$-structure is known as the class $W_3$ (or cocalibrated $G_2$-structure of pure type $W_3$ in some literature), meaning that all the other components of intrinsic torsion except for $W_3$ vanish. For a $G_2$-structure $\varphi$ of class $W_3$, it is characterized [15] by

$$d\varphi \wedge \varphi = 0, \quad d(*_{\varphi} \varphi) = 0,$$

where $*_{\varphi}$ is the Hodge star operator associated to $\varphi$. Notice that the condition $d\varphi \wedge \varphi = 0$ is conformally invariant.
2.4 Conifold Transition

The simplest kind of singularities in algebraic geometry is the so-called *ordinary double point* (ODP), which is modelled on the affine quadric cone \( z_1^2 + \cdots + z_n^2 = 0 \). Obviously such singularities can be resolved by blowing up once. However, Atiyah [8] discovered that the behavior of ODPs in low dimensions is very special. In particular in dimension 3, there exist two small resolutions of ODP that are related by a flop. These small resolutions can be interpreted as blowing up along Weil divisors in algebraic geometry.

Let \( Q \) be the conifold, or in other words the standard affine quadric cone in \( \mathbb{C}^4 \). That is,

\[
Q = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \}.
\]

It is clear that \( Q \) has an isolated ODP at the origin.

By a linear change of coordinates

\[
\begin{align*}
w_1 &= z_1 + iz_2 \\
w_2 &= z_3 + iz_4 \\
w_3 &= iz_4 - z_3 \\
w_4 &= z_1 - iz_2
\end{align*}
\]

we can identify \( Q \) as the zero locus of \( w_1w_4 - w_2w_3 \), or more suggestively,

\[
\det \begin{pmatrix} w_1 & w_2 \\
w_3 & w_4 \end{pmatrix} = 0.
\]

Now let \( \mathbb{C}P^1 \) be parameterized by \( \lambda = [\lambda_1 : \lambda_2] \). Consider

\[
\tilde{Q} = \left\{ (w, \lambda) \in \mathbb{C}^4 \times \mathbb{C}P^1 : \begin{pmatrix} w_1 & w_2 \\
w_3 & w_4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\
\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\
0 \end{pmatrix} \right\}.
\]
It is not hard to see that $\tilde{Q}$ is smooth and the first projection

$$p_1 : \tilde{Q} \to Q$$

is an isomorphism away from $\{0\} \times \mathbb{C}P^1 \subset \tilde{Q}$. Therefore we shall call $\tilde{Q}$ the *resolved conifold* because $p_1 : \tilde{Q} \to Q$ is a small resolution of $Q$ and the exceptional locus $\{0\} \times \mathbb{C}P^1$ is of codimension 2. Moreover, the second projection $p_2 : \tilde{Q} \to \mathbb{C}P^1$ allows us to identify $\tilde{Q}$ with the total space of $\mathcal{O}(-1,-1) \to \mathbb{C}P^1$. Therefore, we see that the resolved conifold $\tilde{Q}$ has trivial canonical bundle.

On the other hand, the ODP in conifold can be easily smoothed out to yield smooth affine quadrics, or the *deformed conifold*

$$Q_t := \left\{ w \in \mathbb{C}^4 : \det \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} = t \right\}.$$  

Clearly $Q_t$ is biholomorphic to the complex Lie group $\text{SL}(2, \mathbb{C})$, which also has trivial canonical bundle.

The geometric transformation

$$\tilde{Q} \to Q \leadsto Q_t$$

is the local model of *conifold transition*. Geometrically the conifold transition can be interpreted as first shrinking a copy of $S^2$ and then replacing it by a copy of $S^3$.

In general, we can start with a Kähler Calabi-Yau 3-fold $\tilde{X}$ with finitely many disjoint $(-1,-1)$-curves, i.e., $\mathbb{C}P^1$’s with $\mathcal{O}(-1,-1)$ as their normal bundles. By blowing down these $(-1,-1)$-curves, we get a singular Calabi-Yau 3-fold $X$ with finitely many ODPs. Under mild assumptions, these ODPs can be smoothed out simultaneously and we get smooth Calabi-Yau’s $X_t$ which are in general non-Kähler [49]. Assuming $\tilde{X}$ is simply connected, by performing conifold transitions described above, we may be able to kill all the $H^2$ of $\tilde{X}$, hence the only nontrivial cohomology group of $X_t$ is $H^3$. By a classification theorem of Wall [114], these non-Kähler Calabi-
Yau 3-folds are diffeomorphic to connected sum of $S^3 \times S^3$’s. In this way, we can construct non-Kähler Calabi-Yau structures on $X_k := \#_k(S^3 \times S^3)$ for $k \geq 2$ [50, 94]. These non-Kähler Calabi-Yau 3-folds are also known to satisfy the $\partial \bar{\partial}$-lemma.

$\tilde{X}$ and $X_t$ are topologically distinct, however, the singular Calabi-Yau $X$ sits on the boundary of the moduli spaces of both $\tilde{X}$ and $X_t$. In this way, Reid [102] conjectured that any two reasonably nice Calabi-Yau 3-folds can be connected via a sequence of conifold transitions, making the moduli space of all nice Calabi-Yau 3-folds connected and reducible.

### 2.5 Hyperkähler Manifolds and Their Twistor Spaces

Let $(N, g)$ be a Riemannian manifold. If in addition $M$ admits three integrable complex structures $I$, $J$ and $K$ with $IJK = -\text{id}$ such that $g$ is a Kähler metric with respect to any of $\{I, J, K\}$, then we call $(N, g, I, J, K)$ a hyperkähler manifold. It turns out that for any $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ satisfying $\alpha^2 + \beta^2 + \gamma^2 = 1$, $g$ is Kähler with respect to the complex structure $\alpha I + \beta J + \gamma K$, therefore we get a $\mathbb{CP}^1$-family of Kähler structures on $N$.

Denote by $\omega_I$, $\omega_J$ and $\omega_K$ the associated Kähler forms with respect to corresponding complex structures. One can easily check that $\omega_I + i \omega_K$ is a holomorphic symplectic $(2,0)$-form with respect to the complex structure $I$, therefore $(N, I)$ has trivial canonical bundle. It also follows that the real dimension of a hyperkähler manifold must be a multiple of 4.

In the real 4-dimension case, if $N$ is compact, then by the Enriques-Kodaira classification of complex surfaces, $N$ must be either a complex torus or a K3 surface. However, if we allow $N$ to be noncompact, there are many more possibilities. An extremely important class of them is the so-called ALE (asymptotically locally Euclidean) spaces. These spaces were first discovered as gravitational (multi-)instantons by physicists [35, 64] and finally classified completely by Kronheimer [87, 88].

It is well-known fact that a hyperkähler 4-manifold is anti-self-dual, therefore its twistor space $Z$ is a complex 3-fold [10]. Roughly speaking, the twistor space
of $N$ is the total space of the $\mathbb{CP}^1$-family of Kähler structures on $N$. Following [78], the twistor space $Z$ of hyperkähler manifolds of arbitrary dimension can be described geometrically as follows. Let $\zeta$ parameterize $\mathbb{CP}^1$. We shall identify $\mathbb{CP}^1$ with $S^2 = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}$ via stereographic projection

$$(\alpha, \beta, \gamma) = \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \frac{\zeta + \bar{\zeta}}{1 + |\zeta|^2}, \frac{i(\bar{\zeta} - \zeta)}{1 + |\zeta|^2}\right).$$

The twistor space $Z$ of $N$ is defined to be the manifold $Z = \mathbb{CP}^1 \times N$ with the almost complex structure $J$ given by

$$J = j \oplus (\alpha I_x + \beta J_x + \gamma K_x)$$

at point $(\zeta, x) \in \mathbb{CP}^1 \times N$, where $j$ is the standard complex structure on $\mathbb{CP}^1$ with holomorphic coordinate $\zeta$. It is a theorem of [78] that $J$ is integrable and the projection $p : Z \to \mathbb{CP}^1$ is a holomorphic fibration (not a holomorphic fiber bundle), which we shall call the holomorphic twistor fibration. Moreover the complex structure $J$ is so twisted that $Z$ does not admit any Kähler structure if $N$ is compact. Let $T^*F$ denote the relative cotangent bundle of the holomorphic twistor fibration $p : Z \to \mathbb{CP}^1$, an important fact is that there exists a global section of $\wedge^2 T^*F \otimes p^*\mathcal{O}(2)$ such that it defines a holomorphic symplectic form on every fiber of $p$.

The twistor spaces of ALE spaces can be described in many other ways. For instance, the twistor spaces of ALE spaces of type $A$ were constructed very concretely using algebraic geometry in [73]. For later use, we shall present a different description of the $A_1$-case here, i.e. the twistor space of the Eguchi-Hansen space, as Hitchin did in [74].

Let $Q$ and $\tilde{Q}$ be the conifold and the resolved conifold described above. Consider the map

$$\rho = z_4 \circ p_1 : \tilde{Q} \xrightarrow{p_1} Q \xrightarrow{z_4} \mathbb{C}.$$ 

It is obvious that, when $z_4 \neq 0$, the fiber $\rho^{-1}(z_4)$ is a smooth affine quadric in $\mathbb{C}^3$. After a little work, we can see that $\rho^{-1}(0)$ is biholomorphically equivalent to $K_{\mathbb{CP}^1}$. 

31
the total space of the canonical bundle of $\mathbb{CP}^1$. It follows that $\rho$ is a fibration.

Now let $\rho' : \tilde{Q}' \to \mathbb{C}$ be another copy of $\rho : \tilde{Q} \to \mathbb{C}$. We may glue these two fibrations holomorphically by identifying $\rho^{-1}(\mathbb{C}^\times) \to \mathbb{C}^\times$ with $\rho'^{-1}(\mathbb{C}^\times) \to \mathbb{C}^\times$ via

$$(z'_1, z'_2, z'_3, z'_4) = \left(\frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4}, \frac{1}{z_4}\right).$$

As a consequence, we get a holomorphic fibration over $\mathbb{CP}^1$, which is exactly the holomorphic twistor fibration of Eguchi-Hansen space.

We conclude that, when performing hyperkähler rotations, there are exactly two complex structures on the Eguchi-Hansen space up to biholomorphism. There is a pair of two antipodal points on $\mathbb{CP}^1$, over which the fibers of the holomorphic twistor fibration are biholomorphic to $K_{\mathbb{CP}^1}$. We shall call these fibers special. All the other fibers are biholomorphic to the smooth affine quadric in $\mathbb{C}^3$. A key observation from this construction is the following proposition.

**Proposition 2.5.1** (Hitchin [74]).

If we remove a special fiber from the total space of the holomorphic twistor fibration of the Eguchi-Hansen space, then what is left is biholomorphic to the resolved conifold $\mathcal{O}(-1, -1)$. 
Chapter 3

The Geometry of the Strominger System

In this chapter, we will study the geometry of the Strominger system from a purely mathematical point of view. Section 3.1 serves as a brief introduction to the Strominger system, with an emphasis on known solutions. As an example, we will present a class of left-invariant solutions to the Strominger system on the complex Lie group $\text{SL}(2, \mathbb{C})$ and its quotients by discrete subgroups in Section 3.2. This work is motivated by understanding the geometry of the deformed conifold. In Section 3.3 we shall explore the relation between solutions to the Strominger system and manifolds with special $G_2$-structure.

3.1 Introduction

Let $X$ be a complex 3-fold with trivial canonical bundle. Being Kähler or not, we shall call such an $X$ a Calabi-Yau 3-fold. Let $\omega$ be a Hermitian metric on $X$ and let $\Omega$ be a nowhere vanishing holomorphic $(3, 0)$-form trivializing $K_X$, the canonical bundle of $X$. In addition, let $(E, h)$ be a holomorphic vector bundle on $X$ equipped with a Hermitian metric.

As we have seen in Chapter 1, the original equations written down by Strominger
(3.1) \[ d^*\omega = d^c \log \|\Omega\|_\omega, \]

(3.2) \[ F \land \omega^2 = 0, \quad F^{0,2} = F^{-2,0} = 0, \]

(3.3) \[ i\partial \bar{\partial} \omega = \frac{\alpha'}{4} (\text{Tr}(R \land R) - \text{Tr}(F \land F)). \]

In the above equations, \( \alpha' \) is a positive coupling constant, while \( R \) and \( F \) are curvature 2-forms of \( T^{1,0}X \) and \( E \) respectively, computed with respect to certain metric connections that we shall further explain. The relevant physical quantities are the flux 3-form

\[ H = \frac{1}{2} d^c \omega \]

and the dilaton field

\[ \phi = \frac{1}{8} \log \|\Omega\|_\omega + \text{constant}. \]

In [103], these equations are derived using local coordinate calculations by imposing \( \mathcal{N} = 1 \) supersymmetry and anomaly cancellation. For a coordinate-free treatment, we refer to Wu’s thesis [115].

Equation (3.1) implies that the reduced holonomy of \( X \) with respect to the Bismut-Strominger connection \( \nabla^b \) is contained in \( SU(3) \). Indeed, by Equation (2.1), we know that

\[ c_1^{\nabla^b}(X) = \frac{1}{2\pi} d \left( d^c \log \|\Omega\|_\omega - d^*\omega \right), \]

which vanishes identically by plugging in Equation (3.1).

The Strominger system was reformulated by Li-Yau [93], where they showed that Equation (3.1) is equivalent to

(3.4) \[ d(\|\Omega\|_\omega \cdot \omega^2) = 0, \]

where \( \|\Omega\|_\omega \) is the norm of \( \Omega \) measured using the Hermitian metric \( \omega \). Li-Yau’s
formulation reveals that if we modify our metric conformally by setting

\[ \tilde{\omega} = \|\Omega\|_\omega^{1/2}\omega, \]

then Equation (3.4) is saying that \( \tilde{\omega} \) is a balanced metric. Since \( X \) admits a balanced metric, we can apply Theorem 2.1.6 when \( X \) is compact. Therefore there are mild topological obstructions to the Strominger system and we can use these obstructions to rule out some non-Kähler Calabi-Yau 3-folds, say certain \( T^2 \)-bundles over Kodaira surface. For this reason, we shall call Equation (3.4) the \textit{conformally balanced equation}. It is soluble if and only if \( X \) admits a balanced metric, which is completely captured by Michelsohn’s theorem.

Equation (3.2) is the \textit{Hermitina-Yang-Mills equation} of degree 0. By a conformal change, we can rewrite it as

\[ F \wedge \tilde{\omega}^2 = 0. \]

Since \( \tilde{\omega} \) is balanced, it is also Gauduchon and we can apply Theorem 2.2.3 to conclude that Equation (3.2) can be solved if and only if the holomorphic vector bundle \( E \) is polystable of degree 0 with respect to the polarization \( \tilde{\omega} \).

The geometry of \( X \) and \( E \) are coupled in the so-called \textit{anomaly cancellation equation} (3.3), which is an equation of \( (2,2) \)-forms. The anomaly cancellation equation (3.3) topologically restricts the second Chern class of \( E \). In addition, if we use Chern connection to compute \( F \), it indicates that \( \text{Tr}(R \wedge R) \) is a \( (2,2) \)-form. Hence from a purely mathematical point of view, the most natural choice of connections on \( T^{1,0}X \) is the Chern connection, as suggested in [103]. However there are physical arguments [79] justifying the use of arbitrary Hermitian connections; while in other literatures (for example [80, 43]), people also add the equation of motion into the system and use the Hull connection to compute the \( \text{Tr}(R \wedge R) \) term. In this thesis, we allow using any Hermitian connection to solve Equation (3.3).

Physically, the Strominger system is derived from the lowest order \( \alpha' \)-expansion of \( \mathcal{N} = 1 \) supersymmetry constraint, therefore a valid torsional heterotic compactification receives higher order \( \alpha' \)-corrections. In this thesis, we will not touch higher
order $\alpha'$-corrections and treat the Strominger system as a closed system.

As a generalization of the flux-free case, solutions to the Strominger system should include Ricci-flat Kähler metrics. Indeed it is the case: by setting $H = \frac{1}{2} d^c \omega = 0$, we conclude that $\omega$ is a Kähler metric and the right hand side of Equation (3.3) vanishes. Then Equation (3.4) implies that $\|\Omega\|_\omega$ is a constant so we have a Ricci-flat Kähler metric. Moreover, we can choose $E = T^{1,0}X$ so $R = F$, therefore Equation (3.3) is satisfied and the Hermitian-Yang-Mills equation (3.2) holds automatically. We will refer to such solutions the Kähler solutions.

In his original paper [103], Strominger described orbifolded solutions and infinitesimal deformations of Kähler solutions. The first irreducible smooth solutions to the Strominger system was constructed by Li and Yau [93]. They considered the case where $X$ is a Kähler Calabi-Yau 3-fold and $E$ is a deformation of the direct sum of $T^{1,0}X$ with trivial bundle. Li and Yau showed that when the deformation is sufficiently small, one can perturb Kähler solutions on $X$ to non-Kähler solutions. Such techniques were further developed in [5, 6] to deal with more general bundles and perturbations.

A breakthrough was due to Fu and Yau. They observed that on the geometric models described by Goldstein-Prokushkin [65] (this is essentially the same construction of Calabi-Eckmann [22]), a clever choice of ansatz reduces the whole Strominger system to a complex Monge-Ampère type equation of a single dilaton function on the Kähler Calabi-Yau 2-fold base. By solving this PDE, Fu and Yau were able to construct mathematically rigorous non-perturbative solutions to the Strominger system, on both compact backgrounds [11, 58, 59] and local models [54]. Such a method can be further modified to yield more heterotic non-Kähler geometries [12]. Fu-Yau’s work has inspired many developments in the analytic theory of the Strominger system, including the form-type Calabi-Yau equations [55, 57], estimates on Fu-Yau equation and its higher dimensional generalization [99, 100, 101], geometric flows leading to solutions of Strominger system [98] etc.

Solutions to the Strominger system have also been found on various nilmanifolds

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1The same ansatz on certain $T^2$-bundles over K3 surfaces was first discussed in [32].
and solvmanifolds [43, 66, 42, 109, 110, 97] and on the blow-up of conifold [27].

To solve the Strominger system, we first need to look for non-Kähler Calabi-Yau 3-folds with balanced metrics. As we have seen in Section 2.4, conifold transition provides us lots of examples of non-Kähler Calabi-Yau 3-folds including $\#_k(S^3 \times S^3)$. Moreover, Fu-Li-Yau [53] showed that the balanced condition is preserved under conifold transition, and the Hermitian-Yang-Mills equation (3.2) is also well-behaved according to the work of Chuan [30, 31]. Therefore it is very tempting to solve the Strominger system on these spaces, especially on $\#_k(S^3 \times S^3)$.

The first step in this direction is to understand the local model of conifold transition. In [24], Candelas-de la Ossa constructed explicit Ricci-flat Kähler metrics on both deformed and resolved conifolds and studied their asymptotic behavior in detail. However, as conifold transitions generally take place in the non-Kähler category, it is desirable to construct non-Kähler solutions to the Strominger system on both deformed and resolved conifolds as well. In this thesis, we will present a class of solutions on the deformed conifold $\text{SL}(2, \mathbb{C})$ in the next section. Solutions on the resolved conifold $\mathcal{O}(-1, -1)$ will be constructed in Chapter 4.

To end this section, let us make a comparison between geometrical models in [25] and [103].

<table>
<thead>
<tr>
<th>Model</th>
<th>Flux</th>
<th>Metric</th>
<th>(3,0)-form</th>
<th>Holonomy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 [25]</td>
<td>$H = 0$</td>
<td>Ricci-flat Kähler</td>
<td>$\nabla^{LC}\Omega = 0$</td>
<td>Hol($\nabla^{LC}$) $\subset$ SU(3)</td>
</tr>
<tr>
<td>2 [25]</td>
<td>$H \neq 0$</td>
<td>balanced</td>
<td>$\nabla^b\Omega = 0$</td>
<td>Hol($\nabla^b$) $\subset$ SU(3)</td>
</tr>
<tr>
<td>3 [103]</td>
<td>$H \neq 0$</td>
<td>conformally balanced</td>
<td>$\nabla^b\Omega \neq 0$</td>
<td>Hol$_0(\nabla^b)$ $\subset$ SU(3)</td>
</tr>
</tbody>
</table>

It is an interesting question to ask whether the existence of Model 2 and Model 3 are equivalent on a given $X$. In terms of Equation (3.4), it is to ask whether the following statement is true or not: If $X$ is a compact Calabi-Yau 3-fold with a balanced metric $\omega_0$, then there exists a balanced metric $\omega$ (preferably in the same cohomology class of $\omega_0$) such that $\|\Omega\|_\omega$ is a constant. This is a balanced version of Calabi (Gauduchon) conjecture and it has been proved by Székelyhidi-Tosatti-Weinkove in [105] under the assumption that $X$ also admits an astheno Kähler metric. Moreover, on $\#_k(S^3 \times S^3)$, these balanced metrics can be characterized as critical points of a
3.2 An Example: Left-invariant Solutions on the Deformed Conifold

In this section, we present a class of left-invariant solutions to the Strominger system on the complex Lie group $\text{SL}(2, \mathbb{C})$, which can also be identified with the deformed conifold. This problem was first considered in [13], where the authors claimed to have constructed such a solution. However, it was pointed out in [7] that the aforementioned solution is not valid. By using the canonical 1-parameter family of Hermitian connections defined in Section 2.2, we are able to construct left-invariant solutions to the Strominger system on $\text{SL}(2, \mathbb{C})$, thus answering a question asked by Andreas and Garcia-Fernandez. Most part of this section has appeared in my joint work with S.-T. Yau [40], with some calculation there simplified.

For simplicity, let us first consider the case where the holomorphic vector bundle $E$ is flat, i.e., $F \equiv 0$. Under such an assumption, the Hermitian-Yang-Mills equation (3.2) is automatically satisfied, hence the Strominger system reduces to the following equations

$$i \partial \bar{\partial} \omega = \frac{\alpha'}{4} \text{Tr} R \wedge R,$$
$$d \left( \| \Omega \| \omega \cdot \omega^2 \right) = 0.$$ 

Let $X$ be a complex Lie group and $e \in X$ be the identity element. Since $X$ is holomorphically parallelizable, it has trivial canonical bundle and we can choose $\Omega$ to be left-invariant. Given any Hermitian metric on $T_eX$, we can translate it to get a left-invariant Hermitian metric $\omega$ on $X$. It follows that with respect to such a metric, $\| \Omega \| \omega$ is a constant and the conformal balanced equation (3.4) indicates that $\omega$ is balanced. The straightforward calculation in [1] shows that $\omega$ is balanced if and only if $X$ is unimodular. In particular this property is independent of the choice of the left-invariant metric $\omega$. 
Now let us assume that $X$ is unimodular and $\omega$ is left-invariant. So Equation (3.4) holds and we only need to deal with the reduced anomaly cancellation equation (3.5).

Let $\mathfrak{g}$ be the complex Lie algebra associated to $X$ and let $e_1, \ldots, e_n \in \mathfrak{g}$ be an orthonormal basis with respect to $\omega$. Let $c^j_{ik} \in \mathbb{C}$ be the structure constants of $\mathfrak{g}$ defined in the usual way
\[
[e_j, e_k] = c^l_{jk} e_l.
\]

Let $\{e^j\}_{j=1}^n$ be the holomorphic 1-forms on $X$ dual to $\{e_1, \ldots, e_n\}$. Then we can express the Hermitian form $\omega$ as
\[
\omega = i \sum_{j=1}^n e^j \wedge \bar{e}^j.
\]

Furthermore, the Maurer-Cartan equation reads
\[
(3.7) \quad \text{d} e^j = -\frac{1}{2} \sum_{k,l} c^j_{kl} e^k \wedge e^l.
\]

Now we shall compute the canonical 1-parameter family of Hermitian connections $\nabla^t$. We may trivialize the holomorphic tangent bundle $T^{1,0}X$ by $\{e_j\}_{j=1}^n$. Under such trivialization, the Chern connection $\nabla^c$ is simply $\text{d}$ and we thus get
\[
\nabla^t = \text{d} + \frac{t}{4} (\text{d}^\omega + \mathcal{M}(\text{d}^\omega)) \triangleq \text{d} + A^t,
\]
where we need to view $A^t$ as an $\text{End}(T^{1,0}X)$-valued 1-form.

By straightforward calculation, we have
\[
(3.8) \quad A^t = \frac{t}{2} \sum_j e^j \otimes \text{ad}(e_j)^T - \bar{e}^j \otimes \overline{\text{ad}(e_j)}.
\]

Consequently,
\[
R^t = \text{d}A^t + A^t \wedge A^t = \frac{t}{2} \sum_j \text{d}e^j \otimes \text{ad}(e_j)^T - \text{d}\bar{e}^j \otimes \overline{\text{ad}(e_j)} + A^t \wedge A^t.
\]
As $\text{Tr}(A^t \wedge A^t) = 0$, it follows directly from unimodularity of $X$ that the first Chern form

$$c_1^{\nabla^t}(X) = \frac{i}{2\pi} \text{Tr}(R^t) = 0.$$ 

It agrees with our prediction since $c_1^{\nabla^t}(X) = c_1^{\nabla^b}(X)$ when the metric is balanced.

Now we want to compute

$$\text{Tr}(R^t \wedge R^t) = \text{Tr}(dA^t \wedge dA^t) + 2\text{Tr}(A^t \wedge A^t \wedge dA^t) + \text{Tr}(A^t \wedge A^t \wedge A^t \wedge A^t).$$

It is a well-known fact that the last term $\text{Tr}(A^t \wedge A^t \wedge A^t \wedge A^t)$ vanishes. Let us compute the first two terms separately.

The first term is

$$\text{Tr}(dA^t \wedge dA^t) = \frac{(t-1)^2}{4} \sum_{j,k} de^j \wedge de^k \cdot \kappa(e_j, e_k) - de^j \wedge d\bar{e}^k \cdot \text{Tr} \left( \text{ad}(e_j)^T \text{ad}(\bar{e}_k) \right)$$

$$+ \text{conjugate of the above line},$$

where $\kappa$ is the Killing form.

It is not hard to see that

$$\sum_{j,k} de^j \wedge de^k \cdot \kappa(e_j, e_k) = 0,$$

hence we conclude

$$\text{Tr}(dA^t \wedge dA^t) = -\frac{(t-1)^2}{2} \sum_{j,k} de^j \wedge d\bar{e}^k \cdot \text{Tr} \left( \text{ad}(e_j)^T \text{ad}(\bar{e}_k) \right).$$

Similarly the second term can be calculated

$$2\text{Tr}(A^t \wedge A^t \wedge dA^t) = -\frac{(t-1)^3}{2} \sum_{j,k} de^j \wedge de^k \cdot \text{Tr} \left( \text{ad}(e_j)^T \text{ad}(e_k) \right),$$

hence

$$\text{Tr}(R^t \wedge R^t) = -\frac{t(t-1)^2}{2} \sum_{j,k} de^j \wedge de^k \cdot \text{Tr} \left( \text{ad}(e_j)^T \text{ad}(e_k) \right).$$
As
\[ i\partial\bar{\partial}\omega = \sum_{j} \, d\bar{e}^j \wedge de^j, \]
the anomaly cancellation equation (3.3) reduces to
\[ \sum_{j} \, de^j \wedge d\bar{e}^j = -\frac{t(t-1)^2}{8} \alpha' \sum_{j,k} \, de^j \wedge d\bar{e}^k \cdot Tr \left( \text{ad}(e_j)^T \text{ad}(e_k) \right). \]

Let \( X = \text{SL}(2, \mathbb{C}) \), so we have proved:

**Theorem 3.2.1.** Let \( \omega \) be the left-invariant Hermitian metric on \( X \) induced by the Killing form, then Equation (3.9) is solvable. By picking \( t < 0 \), for instance the Bismut-Strominger connection, we obtain valid solutions to the Strominger system on \( \text{SL}(2, \mathbb{C}) \).

**Remark 3.2.2.** Because our ansatz is invariant under left translations, solutions to the Strominger system on \( X \) descend to solutions on the quotient \( \Gamma \backslash X \) for any discrete subgroup \( \Gamma \). In particular we get compact models for heterotic superstrings if we choose \( \Gamma \) to be cocompact. There are lots of such \( \Gamma \) coming from hyperbolic 3-manifolds.

Now let us turn to the case that \( E \) is not flat. We may also construct left-invariant solutions to the Strominger system on \( \text{SL}(2, \mathbb{C}) \) in a similar manner.

Let \( \rho : X \to \text{GL}(n, \mathbb{C}) \) be a faithful holomorphic representation, then \( X \) naturally acts on \( \mathbb{C}^n \) from right by setting \( v \cdot g := \overline{\rho(g)^T v} \) for \( g \in X \) which we abbreviate to \( \bar{g}^T v \). Consider the following Hermitian metric \( H \) defined on the trivial bundle \( E = X \times \mathbb{C}^n \): at a point \( g \in X \), the metric is given by
\[ \langle v, w \rangle_g = (v \cdot g)^T \overline{(w \cdot g)} = v^T \bar{g} g^T \bar{w}, \]
where \( v, w \in \mathbb{C}^n \) are arbitrary column vectors. Choose the standard basis for \( \mathbb{C}^n \) as a holomorphic trivialization, then
\[ H_g = (h_{ij})_g = \bar{g} g^T. \]
Let us compute its curvature $F$ with respect to the Chern connection. By the formula $F = \bar{\partial}(\bar{H}^{-1}\partial H)$, we get

$$F = \bar{\partial}[(\bar{g}^T)^{-1}(g^{-1}\partial g)\bar{g}^T]$$

$$= -(\bar{g}^T)^{-1}[(\bar{\partial}g^T \cdot (\bar{g}^T)^{-1})(g^{-1}\partial g) + (g^{-1}\partial g)(\bar{\partial}g^T \cdot (\bar{g}^T)^{-1})]g^T.$$ 

Notice that $g^{-1}\partial g$ is the Maurer-Cartan form

$$g^{-1}\partial g = \sum_j e^j \otimes e_j.$$ 

Therefore

$$F = -(\bar{g}^T)^{-1}\left(\sum_{j,k} e^j \wedge \bar{e}^k \otimes [e_j, \bar{e}_k^T]\right)g^T$$

and thus $\text{Tr}(F) = 0$. Moreover can compute

$$\text{Tr}(F \wedge F) = 2\sum_{j,k} d(e^j \wedge \bar{e}^k \cdot \text{Tr}(e_j \bar{e}_k^T)).$$

Similar calculation shows that the Hermitian-Yang-Mills equation (3.2) is equivalent to

$$(3.10) \quad \sum_j [e_j, \bar{e}_j^T] = 0.$$ 

For $X = \text{SL}(2, \mathbb{C})$, if the Hermitian metric comes from the Killing form, then (3.10) holds and all the three terms in (3.3) are proportional. For $\rho$ is the fundamental representation of $\text{SL}(2, \mathbb{C})$, as along as $t(t - 1)^2 + 1 < 0$, we obtain valid left-invariant solutions to the Strominger system with non-flat $E$.

**Remark 3.2.3.** It is well-known that irreducible $\text{SL}(2, \mathbb{C})$-representations of any dimension can be constructed from taking algebraic operations on the fundamental representation. Therefore using any solutions above, we can produce non-flat solutions to the Strominger system on $\text{SL}(2, \mathbb{C})$ with irreducible $E$ of arbitrary rank.
3.3 Relation with $G_2$-structures

In this section, we shall give a geometric construction of 7-manifolds with $G_2$-structure of class $W_3$ (see Section 2.3) based on a Calabi-Yau 3-fold $X$ satisfying the Hermitian-Yang-Mills equation (3.2) and the conformally balanced equation (3.4). In some sense this is a converse of Calabi-Gray’s construction. Similar idea has already appeared in the work of Chiossi-Salamon [29] and Fernández-Ivanov-Ugarte-Villacampa [44].

Let $(X, \omega, \Omega)$ be a Calabi-Yau 3-fold with Hermitian metric $\omega$ and holomorphic $(3,0)$-form $\Omega = \Omega_1 + i\Omega_2$ satisfying the conformally balanced equation (3.4). From Section 2.3, we may interpret these datum on $X$ coming from a conformal change of a $U(3)$-structure of class $V_3$.

Let $(L, h)$ be a Hermitian holomorphic line bundle over $X$ such that its first Chern form $c_1(L)$ is primitive, i.e.

$$c_1(L) \wedge \omega^2 = 0.$$  \hfill (3.11)

In particular, such an $L$ can be taken to be the determinant line bundle of a holomorphic vector bundle $E$ solving the Hermitian-Yang-Mills equation (3.2). As we have seen in Section 3.1, Equation (3.11) is equivalent to

$$c_1(L) \wedge \tilde{\omega}^2 = 0,$$  \hfill (3.12)

where $\tilde{\omega}$ is a balanced metric conformal to $\omega$. As line bundles are always stable, by the Donaldson-Uhlenbeck-Yau Theorem (2.2.3), we know that Equation (3.12) is solvable if and only if

$$[c_1(L)] \cdot [\tilde{\omega}^2] = 0$$  \hfill (3.13)

as a de Rham cohomology class, which is topological in nature. There are many examples such that Equation (3.13) is satisfied. For example, when $[\tilde{\omega}^2]$ is a rational class and the Picard number of $X$ is at least 2, one can always find such an $L$. 

43
Given such an $L$, let $\bar{M}$ be the total space of the principal $U(1)$-bundle over $M$ associated to $L$. The Chern connection on $L$ gives rise to a globally defined real 1-form $\alpha$ on $\bar{M}$ such that

$$c_1(L) = -\frac{d\alpha}{2\pi}.$$  

We can cook up a $G_2$-structure $\varphi$ on $\bar{M}$ given by

$$\varphi = \Omega_1 - \frac{||\Omega||_\omega}{2\sqrt{2}} \alpha \wedge \omega.$$  

It follows that

$$d\varphi \wedge \varphi = \frac{||\Omega||_\omega}{2\sqrt{2}} \alpha \wedge d\alpha \wedge \omega^2 = 0$$

by Equation (3.11). Let

$$\tilde{\varphi} = \left(\frac{||\Omega||_\omega}{2\sqrt{2}}\right)^{-1/4} \varphi$$

be a conformal change of $\varphi$, then

$$^{*}\tilde{\varphi} \tilde{\varphi} = \Omega_2 \wedge \alpha - \frac{||\Omega||_\omega}{4\sqrt{2}} \omega^2,$$

hence

$$d(^{*}\tilde{\varphi} \tilde{\varphi}) = 0$$

by Equation (3.4) and we get a $G_2$-structure of class $W_3$ on $\bar{M}$. It is easy to see that $(\bar{M}, \tilde{\varphi})$ has holonomy $G_2$ if and only if $X$ is Ricci-flat Kähler and $L$ is flat.

There are not many known constructions of compact 7-manifolds with $G_2$-structures of class $W_3$. The other examples include special Aloff-Wallach manifolds of the form $SU(3)/U(1)$ [16], tangent sphere bundle (gwistor space) over hyperbolic 4-manifolds [3] and geometric models in [44].

Suppose $M$ is simply-connected and $c_1(L) \in H^2(X; \mathbb{Z})$ is not zero, then the above construction yields simply-connected $\bar{M}$ with $G_2$-structure of class $W_3$. A natural question to ask is whether such $\bar{M}$ admits torsion-free $G_2$-structures. One possibility is to look at the Laplacian coflow proposed by Grigorian [69].

In physics language, the above recipe transforms a solution to the 6-dimension
Killing spinor equations on $M$ with arbitrary dilaton to a solution to the 7-dimensional Killing spinor equations on $\bar{M}$. This generalizes the construction presented in [44]. Our construction has the advantage that it transforms geometric objects in $\text{SU}(3)$-geometry into nice geometric objects in $\text{G}_2$-geometry. For example, the famous SYZ conjecture [104] predicts that any Calabi-Yau 3-fold can be realized as a special Lagrangian $T^3$-fibration with singularities. By pulling back such a SYZ fibration to $\bar{M}$, we get a coassociative fibration of $\bar{M}$, which plays an important role in M-theory [70]. Similarly, by pulling back Yang-Mills instantons on $M$, we get the so-called $\text{G}_2$-instantons on $\bar{M}$. 
Chapter 4

A Class of Local Models

The goal of this chapter is to present the construction promised in Theorem A. To motivate our construction, we first study the complex geometry of Calabi-Gray manifolds (cf. Section 2.3) in Section 4.1 and construct degenerate solutions to the Strominger system on them in Section 4.2. In order to understand the degeneracy, we give a new geometrical interpretation of Calabi-Gray manifolds, which leads to a more general construction of non-Kähler Calabi-Yau manifolds. Section 4.3 is devoted to the proof of Theorem A. Some of the materials in this chapter are taken from [37] and [38].

4.1 The Geometry of Calabi-Gray Manifolds

In order to get interesting compactification of heterotic superstrings with flux, as we have seen, one first needs to look for non-Kähler Calabi-Yau 3-folds with balanced metrics. To my knowledge, there are not so many such examples besides those constructed from conifold transitions. For the explicitness of their geometry, Calabi-Gray manifolds are ideal places to start our investigation. In this section, we will study the complex geometry of Calabi-Gray manifolds $M = \Sigma_g \times N$, where $\Sigma_g \subset T^3$ is a minimal surface of genus $g \geq 3$ and $N$ is a hyperkähler 4-manifold. For simplicity, we will mostly restrict ourselves to the case $N = T^4 = \mathbb{C}^2/\Lambda$, where $\Lambda$ is a rank 4 lattice in $\mathbb{C}^2$.

In order to do explicit calculations on $M$, let us first introduce some notations.
Let $e_1, e_2, e_3$ be an orthonormal basis of parallel vector fields on $T^3$ and let $e^1, e^2, e^3$ be the dual 1-forms. Fix $I, J, K$ a set of pairwise anti-commuting complex structures on the hyperkähler manifold $N$, and denote the associated Kähler forms by $\omega_I, \omega_J$ and $\omega_K$ respectively. Let $\zeta : \Sigma_g \to S^2 \subset \mathbb{R}^3$ be the Gauss map and write its components as

$$\zeta(z) = (\alpha(z), \beta(z), \gamma(z)) \in \mathbb{R}^3, \quad z \in \Sigma_g,$$

where $\zeta \in \mathbb{CP}^1$ and $(\alpha, \beta, \gamma) \in S^2$ are related by standard stereographic projection

$$(\alpha, \beta, \gamma) = \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \frac{\zeta + \bar{\zeta}}{1 + |\zeta|^2}, \frac{i(\bar{\zeta} - \zeta)}{1 + |\zeta|^2}\right).$$

Notice that the fundamental 3-form on $\tilde{M} = T^3 \times N$ is given by

$$\varphi = e^1 \wedge \omega_I + e^2 \wedge \omega_J + e^3 \wedge \omega_K - e^1 \wedge e^2 \wedge e^3.$$

It follows that the induced complex structure $J_0$ on $M = \Sigma_g \times N$ is given by

$$J_0 e_1 = -\gamma e_2 + \beta e_3,$$
$$J_0 e_2 = \gamma e_1 - \alpha e_3,$$
$$J_0 e_3 = -\beta e_1 + \alpha e_2,$$
$$J_0 v = \alpha Iv + \beta Jv + \gamma Kv,$$

for arbitrary vector field $v$ tangent to the fibers of $\pi : M \to \Sigma_g$.

The action of $J_0$ on 1-forms can be obtained easily as follow

$$J_0 e^1 = \gamma e^2 - \beta e^3,$$
$$J_0 e^2 = -\gamma e^1 + \alpha e^3,$$
$$J_0 e^3 = \beta e^1 - \alpha e^2,$$
$$J_0 \omega_I = (2\alpha^2 - 1)\omega_I + 2\alpha \beta \omega_J + 2\alpha \gamma \omega_K,$$
$$J_0 \omega_J = 2\alpha \beta \omega_I + (2\beta^2 - 1)\omega_J + 2\beta \gamma \omega_K,$$
$$J_0 \omega_K = 2\alpha \gamma \omega_I + 2\beta \gamma \omega_J + (2\gamma^2 - 1)\omega_K.$$
Denote by $\omega_0$ the induced metric on $M$ from $\tilde{M}$, then

\begin{equation}
\omega_0 = \omega + \alpha \omega_I + \beta \omega_J + \gamma \omega_K
\end{equation}

is balanced according to Theorem 2.3.2, where $\omega$ is the induced Kähler metric on $\Sigma_g$.

Up to now we have not used the fact that $\Sigma_g$ is minimal. Let $f : D \to \Sigma_g \subset \mathbb{R}^3$ given by

$$(u, v) \mapsto (f_1(u, v), f_2(u, v), f_3(u, v))$$

be an isothermal parametrization of $\Sigma_g$ compatible with its orientation. Let $z = u + iv$ and

$$\varphi_j = \frac{\partial f_j}{\partial u} - i \frac{\partial f_j}{\partial v}$$

for $j = 1, 2, 3$. It is a well-known fact that $\Sigma_g$ is a minimal surface is equivalent to that $\varphi_j$ are holomorphic functions and

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0.$$

In addition, the Gauss map $\zeta : \Sigma_g \to \mathbb{CP}^1 = S^2$ is holomorphic.

Setting

$$2\lambda = \varphi_1 \overline{\varphi}_1 + \varphi_2 \overline{\varphi}_2 + \varphi_3 \overline{\varphi}_3,$$

we can easily express $\alpha, \beta, \gamma$ as

$$-2i\lambda \alpha = \varphi_2 \overline{\varphi}_3 - \varphi_3 \overline{\varphi}_2,$$

$$-2i\lambda \beta = \varphi_3 \overline{\varphi}_1 - \varphi_1 \overline{\varphi}_3,$$

$$-2i\lambda \gamma = \varphi_1 \overline{\varphi}_2 - \varphi_2 \overline{\varphi}_1.$$

Without too much effort, one can check that

$$\varphi_1^{-1} \frac{\partial \alpha}{\partial \bar{z}} = \varphi_2^{-1} \frac{\partial \beta}{\partial \bar{z}} = \varphi_3^{-1} \frac{\partial \gamma}{\partial \bar{z}}.$$
We also have the relations

\[-i \frac{\partial \alpha}{\partial \bar{z}} = \beta \frac{\partial \gamma}{\partial \bar{z}} - \gamma \frac{\partial \beta}{\partial \bar{z}}\]
\[-i \frac{\partial \beta}{\partial \bar{z}} = \gamma \frac{\partial \alpha}{\partial \bar{z}} - \alpha \frac{\partial \gamma}{\partial \bar{z}},\]
\[-i \frac{\partial \gamma}{\partial \bar{z}} = \alpha \frac{\partial \beta}{\partial \bar{z}} - \beta \frac{\partial \alpha}{\partial \bar{z}}.\]  

(4.3)

Now let us assume that \(N = T^4\) and let \(e^4, e^5, e^6, e^7\) be a set of parallel orthonormal 1-forms on \(T^4\) such that

\[\omega_I = e^4 \wedge e^5 + e^6 \wedge e^7,\]
\[\omega_J = e^4 \wedge e^6 - e^5 \wedge e^7,\]
\[\omega_K = e^4 \wedge e^7 + e^5 \wedge e^6.\]

In terms of this frame, it is straightforward to write down the holomorphic (3,0)-form \(\Omega = \Omega_1 + i \Omega_2\) where

\[\Omega_1 = e^1 \wedge \omega_I + e^2 \wedge \omega_J + e^3 \wedge \omega_K,\]
\[\Omega_2 = (-\gamma e^2 + \beta e^3) \wedge \omega_I + (\gamma e^1 - \alpha e^3) \wedge \omega_J + (-\beta e^1 + \alpha e^2) \wedge \omega_K.\]

For later calculation of curvature form and Chern classes, it is convenient to solve for a local holomorphic frame on \((M, J_0)\). Compared to holomorphic vector fields, it is easier to work with holomorphic 1-forms.

Consider a (1,0)-form \(\theta\) of the form

\[\theta = Ldz + Ae^4 + Be^5 + Ce^6 + De^7,\]

where \(z\) is a local holomorphic coordinate on \(\Sigma_g\) and \(L, A, B, C, D\) are complex-valued smooth functions to be determined.
As $J_0\theta = i\theta$, it follows that

\[ iA = \alpha B + \beta C + \gamma D, \]
\[ iB = -\alpha A + \gamma C - \beta D, \]
\[ iC = -\beta A - \gamma B + \alpha D, \]
\[ iD = -\gamma A + \beta B - \alpha C. \]

Solving $A$ and $B$ from $C$, $D$, we get

\[
A = -\frac{\alpha \gamma + i\beta}{\beta^2 + \gamma^2} C + \frac{\alpha \beta - i\gamma}{\beta^2 + \gamma^2} D := -\kappa C + \sigma D, \\
B = \frac{\alpha \beta - i\gamma}{\beta^2 + \gamma^2} C + \frac{\alpha \gamma + i\beta}{\beta^2 + \gamma^2} D := \sigma C + \kappa D, 
\]

where

\[ \kappa = \frac{\alpha \gamma + i\beta}{\beta^2 + \gamma^2} = \frac{i}{2} \left( \zeta + \frac{1}{\zeta} \right) \text{ and } \sigma = \frac{\alpha \beta - i\gamma}{\beta^2 + \gamma^2} = -\frac{1}{2} \left( \zeta - \frac{1}{\zeta} \right) \]

are (local) holomorphic functions on $\Sigma_g$.

If $\theta$ is a holomorphic $(1,0)$-form, then

\[ d\theta = dL \wedge dz + dA \wedge e^4 + dB \wedge e^5 + dC \wedge e^6 + dD \wedge e^7 \]

is of type $(2,0)$, which is equivalent to that

\[ J_0(d\theta) = -d\theta. \]

As a consequence, we have

\[ (dA + \alpha J_0 dB + \beta J_0 dC + \gamma J_0 dD) \wedge e^4 + (dB - \alpha J_0 dA + \gamma J_0 dC - \beta J_0 dD) \wedge e^5 + (dC - \beta J_0 dA - \gamma J_0 dB + \alpha J_0 dD) \wedge e^6 + (dD - \gamma J_0 dA + \beta J_0 dB - \alpha J_0 dC) \wedge e^7 + 2\partial L \wedge dz = 0. \]
Plugging in (4.4), we get

\[2\bar{\partial}L \wedge dz + 2\bar{\partial}C \wedge (-\kappa e^4 + \sigma e^5 + e^6) + 2\bar{\partial}D \wedge (\sigma e^4 + \kappa e^5 + e^7)\]
\[+(C\partial\sigma + D\partial\kappa) \wedge (i\alpha e^4 + e^5 - i\gamma e^6 + i\beta e^7)\]
\[+(C\partial\kappa - D\partial\sigma) \wedge (-e^4 + i\alpha e^5 + i\beta e^6 + i\gamma e^7)\]
\[=0.\]

Each term in the above equation is a (1,1) form. Notice that

\[\{dz, -\kappa e^4 + \sigma e^5 + e^6, \sigma e^4 + \kappa e^5 + e^7\}\]

form a basis for (1,0)-forms, so we deduce that \(\bar{\partial}C = \bar{\partial}D = 0\) and

\[2\bar{\partial}L = (C\sigma_z + D\kappa_z)(i\alpha e^4 + e^5 - i\gamma e^6 + i\beta e^7)\]
\[+(C\kappa_z - D\sigma_z)(-e^4 + i\alpha e^5 + i\beta e^6 + i\gamma e^7),\]

which is always locally solvable since the right hand side is \(\bar{\partial}\)-closed.

Therefore we conclude that

\[\{dz, L_1 dz - \kappa e^4 + \sigma e^5 + e^6, L_2 dz + \sigma e^4 + \kappa e^5 + e^7\}\]

is a local holomorphic frame of \((T^*)^{1,0}M\), where \(L_1\) and \(L_2\) are functions satisfying

\[2\bar{\partial}L_1 = \sigma_z(e^5 + iJ_0 e^5) - \kappa_z(e^4 + iJ_0 e^4) = \frac{2i\alpha_z}{\beta^2 + \gamma^2}(e^7 + iJ_0 e^7)\]
\[= -\frac{i\zeta_z}{\zeta}(e^7 + iJ_0 e^7),\]

\[2\bar{\partial}L_2 = \kappa_z(e^5 + iJ_0 e^5) + \sigma_z(e^4 + iJ_0 e^4) = -\frac{2i\alpha_z}{\beta^2 + \gamma^2}(e^6 + iJ_0 e^6)\]
\[= \frac{i\zeta_z}{\zeta}(e^6 + iJ_0 e^6)\].

After taking dual basis and rescaling, we obtain a holomorphic frame of \(T^{1,0}M\) as
follows
\[
V_1 = i\beta e_4 + i\gamma e_5 + e_6 - i\alpha e_7 = e_6 - iJ_0 e_6,
\]
(4.6)
\[
V_2 = i\gamma e_4 - i\beta e_5 + i\alpha e_6 + e_7 = e_7 - iJ_0 e_7,
\]
\[
V_0 = 2\frac{\partial}{\partial z} - L_1 V_1 - L_2 V_2.
\]

Observe that $V_1$ and $V_2$ are globally defined and nowhere vanishing. Similarly $e_4 - iJ_0 e_4$ and $e_5 - iJ_0 e_5$ are nowhere vanishing holomorphic vector fields on $M$. This should not be surprising, since by our description of $J_0$, translations on $T^4$ are holomorphic automorphisms of $M$, and they generate 4 linearly independent global holomorphic vector fields.

At point where $(\alpha, \beta, \gamma) = (1, 0, 0)$, we have $V_1 + iV_2 = 0$. Similarly at point where $(\alpha, \beta, \gamma) = (-1, 0, 0)$, we see $V_1 - iV_2 = 0$. Notice that the Gauss map $\zeta$ is surjective, so we conclude that as holomorphic vector fields, both $V_1 + iV_2$ and $V_1 - iV_2$ have zeroes.

In [91], LeBrun and Simanca proved that on a compact Kähler manifold, the set of holomorphic vector fields with zeroes is actually a vector space. Hence we obtain a different proof that $M$ is non-Kähler. In fact, we can prove a little more:

**Proposition 4.1.1.**

All the holomorphic $(1,0)$-forms are pullbacks from $\Sigma_g$, therefore $h^{1,0}(M) = h^{1,0}(\Sigma_g) = g$.

**Proof.** Let $\xi$ be a holomorphic $(1,0)$-form on $M$. Notice that $e_j - iJ_0 e_j$ is a holomorphic vector field on $M$ for $j = 4, 5, 6, 7$, so

\[
c_j := \xi(e_j - iJ_0 e_j)
\]

is a holomorphic function on $M$, hence a constant. On the other hand, since

\[
e_4 - iJ_0 e_4 + i\alpha(e_5 - iJ_0 e_5) + i\beta(e_6 - iJ_0 e_6) + i\gamma(e_7 - iJ_0 e_7) = 0,
\]

53
we have

\[ c_4 + i\alpha c_5 + i\beta c_6 + i\gamma c_7 = 0. \]

The only possibility is that \( c_4 = c_5 = c_6 = c_7 = 0 \), otherwise we have a nontrivial relation between \( \alpha, \beta \) and \( \gamma \), which contradicts the fact that the Gauss map \( \zeta \) is surjective.

Now let \( z \) be a local holomorphic coordinate on \( U \subset \Sigma_g \). Then on \( U \times T^4 \subset M \), \( \xi \) can be written as \( \xi = fdz \) for some smooth function \( f \) defined on \( U \times T^4 \). Since \( \xi \) is holomorphic, we know that \( \bar{\partial}f = 0 \) on \( U \times T^4 \), hence \( f \) is a constant on each fiber of \( p : M \to \Sigma_g \). Consequently \( \xi \) is a pullback of holomorphic \((1, 0)\)-form from \( \Sigma_g \).

**Corollary 4.1.2.**

\( M \) does not satisfy the \( \partial\bar{\partial} \)-lemma, hence it is not of Fujiki class \( \mathcal{C} \).

**Proof.** On one hand we have seen that \( h^{1,0}(M) = g \). On the other hand, we know that \( h^{1,0}(M) + h^{0,1}(M) \geq b_1(M) = 2g + 4 \). Therefore \( h^{0,1}(M) \geq g + 4 > g = h^{1,0}(M) \) and the \( \partial\bar{\partial} \)-lemma fails.

In fact a \((g + 4)\)-dimension subspace of \( H^{0,1}(M) \) can be constructed explicitly as the span of pullback of \( H^{0,1}(\Sigma_g) \) and \( e^j + i J_0 e^j \) for \( j = 4, 5, 6, 7 \).

It was conjectured in [47] that if a compact complex manifold admits both balanced and pluriclosed metrics (a priori they are different), then it must be Kähler. This conjecture has been solved in a few cases, including connected sums of \( S^3 \times S^3 \) [53], twistor spaces of anti-self-dual 4-manifolds [112], manifolds of Fujiki class \( \mathcal{C} \) [28], nilmanifolds and certain solvmanifolds [47, 48].

To verify this conjecture for our \( M \), we prove that

**Theorem 4.1.3.**

\( M \) does not admit any pluriclosed metrics. Notice that \( M \) is not of Fujiki class \( \mathcal{C} \), so our theorem is not covered by Chiose’s result [28].

**Proof.** Let \( \rho^j = e^j - i J_0 e^j \) for \( j = 4, 5, 6, 7 \). Clearly they are \((1,0)\)-forms on \( M \). Observe that

\[ d\rho^j = -i d(J_0 e^j) \]
is purely imaginary. On the other hand, $d\rho^j$ is of type $(2,0)+(1,1)$, therefore we conclude that $\partial \rho^j = 0$ and

$$\bar{\partial} \rho^j = -id(J_0 e^j).$$

Assume that $M$ admits a pluriclosed metric $\omega'$, then by integration by part, we have

$$\int_M (d(J_0 \rho^j))^2 \wedge \omega' = \int_M \bar{\partial} \rho^j \wedge \partial \bar{\rho}^j \wedge \omega' = \int_M \rho^j \wedge \bar{\rho}^j \wedge \partial \bar{\partial} \omega' = 0.$$

On the other hand, explicit calculation shows that

$$\sum_{j=4}^7 (d(J_0 \rho^j))^2 = -4d\beta \wedge d\gamma \wedge \omega_I - 4d\gamma \wedge d\alpha \wedge \omega_J - 4d\alpha \wedge d\beta \wedge \omega_K.$$

Observe that

$$\frac{d\beta \wedge d\gamma}{\alpha} = \frac{d\gamma \wedge d\alpha}{\beta} = \frac{d\alpha \wedge d\beta}{\gamma} = \frac{id\zeta \wedge d\bar{\zeta}}{(1+|\zeta|^2)^2} = \zeta^* \omega_{\mathbb{CP}^1}$$

is the pullback of the Fubini-Study metric by the Gauss map $\zeta$. Therefore we have

$$0 = \sum_{j=4}^7 \int_M (d(J_0 \rho^j))^2 \wedge \omega' = -4 \int_M \zeta^* \omega_{\mathbb{CP}^1} \wedge (\alpha \omega_I + \beta \omega_J + \gamma \omega_K) \wedge \omega'.$$

This is in contradiction with the positivity of $\omega'$, therefore $M$ does not admit any pluriclosed metrics, which answers a question of Fu-Wang-Wu [56].

4.2 Degenerate Solutions on Calabi-Gray Manifolds

Recall from Equation (4.2) that the naturally induced metric

$$\omega_0 = \omega + \alpha \omega_I + \beta \omega_J + \gamma \omega_K$$

is balanced and $||\Omega||_{\omega_0} = \text{constant}$, therefore it solves the conformally balanced equation (3.4). However, this metric does not solve the Hermitian-Yang-Mills equation (3.2) and therefore some modifications are needed.
Let $f$ be any real-valued smooth function on $\Sigma_g$. We can cook up a new metric

$$\omega_f = e^{2f} \omega + e^f (\alpha \omega_I + \beta \omega_J + \gamma \omega_K).$$

Obviously

$$\|\Omega\|_{\omega_f} = e^{-2f} \|\Omega\|_{\omega_0}$$

and

$$\omega_f^2 = 2e^{3f} \omega \wedge (\alpha \omega_I + \beta \omega_J + \gamma \omega_K) + 2e^{2f} e^4 \wedge e^5 \wedge e^6 \wedge e^7.$$ 

It follows that $\omega_f$ always solves the conformally balanced equation

$$d(\|\Omega\|_{\omega_f} \omega_f^2) = 0.$$ 

Following the idea of [59], in order to solve the Strominger system on $M$, we can use the ansatz $\omega_f$ as our metric and we are allowed to vary $f$ freely to solve the other two equations.

Let us first look at the anomaly cancellation equation (3.3).

Since have worked out a local holomorphic frame of $M$ in Section 4.1, we can easily compute the term $\text{Tr}(R_f \wedge R_f)$ in (3.3), with respect to the Chern connection associated to $\omega_f$.

With respect to the local holomorphic frame $\{V_0, V_1, V_2\}$, the metric $\omega_f$ is given by the matrix

$$H = 2e^f \begin{pmatrix} e^f \lambda + |L_1|^2 + |L_2|^2 - i\alpha (L_1 \bar{L}_2 - L_2 \bar{L}_1) & -L_1 - i\alpha L_2 & -L_2 + i\alpha L_1 \\ -\bar{L}_1 + i\alpha \bar{L}_2 & 1 & -i\alpha \\ -\bar{L}_2 - i\alpha \bar{L}_1 & i\alpha & 1 \end{pmatrix}.$$ 

The inverse matrix can be computed accordingly

$$H^{-1} = \frac{1}{2e^{2f}\lambda} \begin{pmatrix} 1 & L_1 & L_2 \\ \bar{L}_1 & |L_1|^2 & L_2 \bar{L}_1 \\ \bar{L}_2 & L_1 \bar{L}_2 & |L_2|^2 \end{pmatrix} + \frac{1}{2e^f(1 - \alpha^2)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i\alpha \\ 0 & -i\alpha & 1 \end{pmatrix}.$$ 

56
Let
\[ p = e^{2f\lambda}, \]
\[ R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \]
and set
\[ U = \begin{pmatrix} -L_1 & -L_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -L \\ \text{id} \end{pmatrix}, \]
\[ S = e^f \begin{pmatrix} 1 & -i\alpha \\ i\alpha & 1 \end{pmatrix}, \]
\[ V = \begin{pmatrix} 1 \\ \bar{L}_1 \\ \bar{L}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \bar{L}^T \end{pmatrix}. \]

We can express \( H \) and \( H^{-1} \) as
\[ H = 2pR + 2US\bar{U}^T, \]
\[ H^{-1} = \frac{1}{2p}V\bar{V}^T + \frac{1}{2} \begin{pmatrix} 0 \\ S^{-1} \end{pmatrix}. \]

Direct computation shows that
\[ \bar{H}^{-1}\partial\bar{H} = (p^{-1}\partial p) \begin{pmatrix} 1 & 0 \\ \bar{L}^T & 0 \end{pmatrix} + p^{-1} \begin{pmatrix} \partial\bar{L} \\ \bar{L}^T \partial\bar{L} \end{pmatrix} \begin{pmatrix} \bar{S}L^T & -\bar{S} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{S}^{-1}\partial\bar{S} \end{pmatrix} \begin{pmatrix} L^T & \text{id} \end{pmatrix} \]
\[ - \begin{pmatrix} 0 & 0 \\ \partial L^T & 0 \end{pmatrix}. \]
As a consequence, we have

\[ R_f = \bar{\partial} (\bar{H}^{-1} \partial H) \]
\[ = (\bar{\partial} \log p) \begin{pmatrix} 1 & 0 \\ L^T & 0 \end{pmatrix} - \partial \log p \wedge \begin{pmatrix} 0 & 0 \\ \partial L^T & 0 \end{pmatrix} - \frac{\bar{\partial} p}{p^2} \begin{pmatrix} \partial L \\ L^T \partial \bar{L} \end{pmatrix} \begin{pmatrix} \bar{S} L^T & -\bar{S} \end{pmatrix} + p^{-1} \left( \frac{\partial \bar{L}}{\partial (L^T \partial \bar{L})} \right) \begin{pmatrix} S L^T & -\bar{S} \end{pmatrix} - p^{-1} \left( \frac{\partial L}{L^T \bar{L}} \right) \left( \bar{\partial} (S L^T) - \bar{\partial} \bar{S} \right) \]
\[ + \left( \begin{pmatrix} 0 \\ \bar{\partial} (S^{-1} \partial \bar{S}) \end{pmatrix} \right) (-L^T \text{ id}) + \left( \begin{pmatrix} 0 \\ \bar{S}^{-1} \partial \bar{S} \end{pmatrix} \right) \left( \bar{\partial} L^T \ 0 \right) - \left( \begin{pmatrix} 0 \\ \bar{\partial} \partial L^T \end{pmatrix} \right) \right]. \]

From this we see immediately that

\[ \text{Tr}(R_f) = 4 \bar{\partial} \partial f. \]

An even more complicated calculation reveals that

\[ \text{Tr}(R_f \wedge R_f) \]
\[ = 2 \left[ -\frac{\bar{\partial} \log p}{p} \partial L \cdot \bar{S} \cdot \bar{\partial} L^T - \frac{\partial \log p \wedge \bar{\partial} p}{p^2} \partial L \cdot \bar{S} \cdot \partial L^T + \frac{\partial \log p}{p} \bar{\partial} L \cdot \bar{S} \cdot \bar{\partial} L^T \right] - \frac{\partial \log p}{p} \partial L \cdot \bar{S} \cdot \bar{\partial} L^T + \frac{\bar{\partial} p}{p^2} \partial L \cdot \bar{S} \cdot \partial L^T - \frac{\bar{\partial} p}{p^2} \partial L \cdot \bar{S} \cdot \bar{\partial} L^T + \frac{1}{p} \partial L \cdot \bar{S} \bar{\partial} (S^{-1} \partial \bar{S}) \bar{\partial} L^T \]
\[ - \frac{1}{p} \bar{\partial} \partial L \cdot \bar{S} \cdot \bar{\partial} L^T + \frac{1}{p} \partial \bar{\partial} L \cdot \bar{S} \cdot \bar{\partial} L^T + \frac{1}{p} \partial L \cdot \bar{S} \cdot \bar{S}^{-1} \bar{\partial} \bar{S} \cdot \bar{\partial} L^T - \frac{1}{p} \partial L \cdot \bar{S} \cdot \bar{\partial} L^T \].

Let \( W = \partial \bar{L} \cdot \bar{S} \bar{\partial} L^T \). After a recombination of terms, we get a very simple expression

\[ \text{Tr}(R_f \wedge R_f) \]
\[ = - \frac{2}{p} \left( (\bar{\partial} \partial \log p + \partial \log p \wedge \bar{\partial} \log p) W - \partial \log p \wedge \bar{\partial} W + \bar{\partial} log p \wedge \partial W - \bar{\partial} \partial W \right) \]
\[ = 2 \bar{\partial} \partial \left( \frac{W}{p} \right). \]

Recall that \( \bar{\partial} L \) can be read off from (4.5), hence we are able to calculate this term.
explicitly as
\[
\frac{W}{p} = -\frac{2i}{e^f \lambda} \frac{\lvert \zeta \rvert^2}{(1 + \lvert \zeta \rvert^2)^2} (\alpha \omega_I + \beta \omega_J + \gamma \omega_K) = -\frac{i}{4e^f} \lVert d\zeta \rVert^2 (\alpha \omega_I + \beta \omega_J + \gamma \omega_K),
\]
where \( \zeta : \Sigma_g \rightarrow \mathbb{CP}^1 \) is the Gauss map. Clearly this term is globally defined.

A crucial consequence of the lengthy calculation above is that \( \text{Tr}(R_f \wedge R_f) \) is \( \partial \bar{\partial} \)-exact. Therefore it is possible to set \( F \equiv 0 \), i.e. \( E \) is flat, to solve the Hermitian-Yang-Mills (3.2) without violating the cohomological restriction in (3.3).

We also observe that
\[
i\partial \bar{\partial} \omega_f = i\partial \bar{\partial} (e^f (\alpha \omega_I + \beta \omega_J + \gamma \omega_K)).
\]
Therefore by equating
\[
e^{2f} = \frac{\alpha'}{8} \lVert d\zeta \rVert^2,
\]
we solve the whole Strominger system with \( F \equiv 0 \).

Unfortunately \( \zeta : \Sigma_g \rightarrow \mathbb{CP}^1 \) is a branched cover of degree \( g - 1 \), therefore \( \lVert d\zeta \rVert^2 \) vanishes at the ramification points. At these ramification points \( f \) goes to \( -\infty \), thus the metric \( \omega_f \) is degenerate at the fibers of \( \pi : M \rightarrow \Sigma_g \) over these ramification points. So what we really get is a degenerate solution to the Strominger system.

To understand the degeneracies, we have the following key observation.

Comparing the complex structures on \( M = \Sigma_g \times N \) and the twistor space \( Z \) of \( N \), we observe that
\[
\begin{array}{ccc}
  (M, J_0) & \cong & \zeta^* Z \\
\downarrow \pi & & \downarrow \rho \\
  \Sigma_g & \cong & \mathbb{CP}^1 \\
\end{array}
\]
is a pullback square! In other words, Calabi-Gray manifolds can be identified with the total space of pullback of the holomorphic twistor fibration of \( Z \) over \( \mathbb{CP}^1 \) via the Gauss map of minimal surfaces \( \Sigma_g \) in \( T^3 \).

With the pullback picture understood, we can immediately generalize Calabi-Gray’s construction as follows.
Let $N$ be a hyperkähler manifold of complex dimension $2n$ and let $p : Z \to \mathbb{CP}^1$ be its holomorphic twistor fibration. Suppose $h : Y \to \mathbb{CP}^1$ is a holomorphic map and let $\tilde{Y} = h^*Z$ be the total space of the holomorphic twistor fibration. By a simple Chern class calculation, one deduces that

$$K_{\tilde{Y}} \cong K_Y \otimes h^*\mathcal{O}(-2n).$$

Therefore we have

**Theorem 4.2.1.**

Given a compact complex manifold $Y$ with $h : Y \to \mathbb{CP}^1$ is a nonconstant holomorphic map such that

$$(4.7) \quad K_Y \cong h^*\mathcal{O}(2n),$$

then $\tilde{Y}$ constructed above is a non-Kähler Calabi-Yau manifold. Moreover, $\tilde{Y}$ admits a balanced metric if and only if $Y$ does so.

A similar construction was used by LeBrun [89] for different purposes.

*Proof.* The above calculations show that once (4.7) is satisfied, then $K_{\tilde{Y}}$ is trivial.

Let us assume that $\tilde{Y}$ is Kähler, then $Y$ is also Kähler since as a smooth manifold $\tilde{Y} = Y \times N$ and $Y \times \{\text{pt}\}$ is a section of the holomorphic fibration $\pi : \tilde{Y} \to Y$ for any $\{\text{pt}\} \in N$. On one hand, by Yau’s theorem [116, 118], $\tilde{Y}$ admits a Ricci-flat Kähler metric. On the other hand, since $h : Y \to \mathbb{CP}^1$ is not a constant, we know that $K_Y = h^*\mathcal{O}(2n)$ is nonnegative and $c_1(Y)$ can be represented by a negative semi-definite $(1,1)$-form which is not identically 0. By Yau’s theorem again, $Y$ admits a Kähler metric whose Ricci curvature is nonpositive and negative somewhere. Therefore, we have a nonconstant holomorphic map $\pi : \tilde{Y} \to Y$ from a compact Ricci-flat Kähler manifold to a negatively-curved compact manifold, which contradicts with Yau’s generalized Schwarz lemma [117]. Therefore $\tilde{Y}$ cannot be Kähler.

If $\tilde{Y}$ is balanced, it follows from a theorem of Michelsohn [96] that $Y$ must be balanced. Conversely, if $\omega$ is a balanced metric on $Y$, then we can write down an
explicit balanced metric $\omega_0$ on $\tilde{Y}$, using the expression (4.2).

If (4.7) is satisfied, then $L = h^*\mathcal{O}(n)$ is a square root of $K_Y$, which corresponds to a spin structure on $Y$ according to Atiyah [9]. $L$ is known as a theta characteristic in the case that $Y$ is a complex curve. The minimal surface $\Sigma_g$ in a Calabi-Gray manifold is a special case of the above construction with $n = 1$. For $Y$ a curve and $n = 1$, such an $h$ exists if and only if there is a theta characteristic $L$ on $Y$ such that $h^0(Y, L) \geq 2$, i.e., $L$ is a vanishing theta characteristic.

**Example 4.2.2.** For every hyperelliptic curves $Y$ of genus $g \geq 3$, vanishing theta characteristics exist, so Theorem 4.2.1 can be used to construct non-Kähler balanced Calabi-Yau 3-folds. However, it is a theorem of Meeks [95] that if $g$ is even, $Y$ cannot be minimally immersed in $T^3$. From this we see that Theorem 4.2.1 yields examples not covered by Calabi-Gray.

Actually, the set of genus $g$ curves with a vanishing theta characteristic defines a divisor in the moduli space of genus $g$ curves. More refined results of this type can be found in [71] and [106].

**Example 4.2.3.** If we allow $Y$ to be of higher dimension, then Theorem 4.2.1 can be used to construct simply-connected non-Kähler Calabi-Yau manifolds of higher dimension. For instance, we can take $Y \subset \mathbb{CP}^1 \times \mathbb{CP}^r$ to be a smooth hypersurface of bidegree $(2n + 2, r + 1)$, then (4.7) is satisfied, where $h$ is the restriction of the projection to $\mathbb{CP}^1$. There are also numerous examples of elliptic fibrations over $\mathbb{CP}^1$ without multiple fibers such that (4.7) holds.

### 4.3 Construction of Local Models

In last section, we constructed degenerate solutions to the Strominger system on Calabi-Gray manifolds and we see that the degeneracy occurs exactly at the fibers over branching locus of the Gauss map. Since Calabi-Gray manifolds can be identified with the pullback of the holomorphic twistor fibration via the Gauss map, if we consider the Strominger system on the twistor space itself, then we no longer have the problem.
of degeneracies. However, a twistor space can never have trivial canonical bundle, therefore for the Strominger system to make sense, we need to remove a divisor from the twistor space to make it a noncompact Calabi-Yau.

Let $N$ be a hyperkähler 4-manifold and $p : Z \to \mathbb{C}P^1$ be its holomorphic twistor fibration. Let $F$ be an arbitrary fiber of $p$. Without loss of generality, we may assume that $F$ is the fiber over $\infty \in \mathbb{C}P^1$. Let $X = Z \setminus F$, then $X$ is a noncompact Calabi-Yau 3-fold, since we can write down a holomorphic $(3,0)$-form explicitly as

$$\Omega := (-2\zeta \omega_I + (1 - \zeta^2)\omega_J + i(1 + \zeta^2)\omega_K) \wedge d\zeta,$$

where as before, $\omega_I$, $\omega_J$ and $\omega_K$ are Kähler forms on $N$ and $\zeta \in \mathbb{C}$ parameterizes $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$.

In this case, we still have a fibration structure over $\mathbb{C}$:

$$\begin{align*}
X = Z \setminus F &\longrightarrow Z \\
p &\downarrow \quad p \\
C &\longrightarrow \mathbb{C}P^1
\end{align*}$$

When $N$ is $\mathbb{C}^2$ with standard hyperkähler metric, $X$ constructed above is biholomorphic to $\mathbb{C}^3$. If $N$ is the Eguchi-Hansen space with $F$ chosen to be special, then according to Hitchin (see Section 2.5), $X$ is biholomorphic to the resolved conifold $\mathcal{O}(-1, -1)$.

In this section, we shall present explicit solutions to the Strominger system on above constructed $X$ for any hyperkähler 4-manifold $N$. In particular, we get important local models of solutions on $\mathbb{C}^3$ and $\mathcal{O}(-1, -1)$. Hopefully these solutions can be used for gluing in future investigations.

Our strategy will be very similar to what we did in the Calabi-Gray case. We will first write down an ansatz solving the conformally balanced equation (3.4), which depends on certain functions. Then we tune the functions to solve the whole Strominger system. Notice that the curvature of $N$ plays an important role in this section, which guides us to a natural choice of the holomorphic vector bundle $E$. However, the price
to pay is that all the calculations are much more complicated.

Again, let us start with the conformally balanced equation (3.4). Observe that $X$ is diffeomorphically a product $\mathbb{C} \times N$ with twisted complex structure. Let $h : N \to \mathbb{R}$ and $g : \mathbb{C} \to \mathbb{R}$ be arbitrary smooth functions. In addition, we use

$$\omega_{\mathbb{CP}^1} = \frac{2i}{(1 + |\zeta|^2)^2} d\zeta \wedge d\bar{\zeta}$$

to denote the round metric of radius 1 on $\mathbb{CP}^1$ and its restriction on $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$.

Now consider the Hermitian metric

$$(4.8) \quad \omega = \frac{e^{2h+g}}{(1 + |\zeta|^2)^2} (\alpha \omega_I + \beta \omega_J + \gamma \omega_K) + e^{2g} \omega_{\mathbb{CP}^1}$$

on $X = \mathbb{C} \times N$. One can check that

$$(4.9) \quad \|\Omega\|_\omega = c \cdot \frac{(1 + |\zeta|^2)^4}{e^{2h+2g}}$$

for some positive constant $c$ and

$$(4.10) \quad \omega^2 = 2 e^{4h+2g} \text{vol}_N + 2 e^{2h+3g} (\alpha \omega_I + \beta \omega_J + \gamma \omega_K) \wedge \omega_{\mathbb{CP}^1},$$

where $\text{vol}_N$ is the volume form on $N$. It follows that $\omega$ solves the conformally balanced equation (3.4) for arbitrary $g$ and $h$ by direct computation.

Now we proceed to solve the anomaly cancellation equation (3.3) using ansatz (4.8). The first step would be to compute the curvature term $\text{Tr}(R \wedge R)$, using the Chern connection, with respect to the metric (4.8). To do so, following the method we used in last section, it is convenient to first solve for a local holomorphic frame of $(1,0)$-forms on $X$.

We fix $I$ to be the background complex structure on the hyperkähler 4-manifold $N$. Locally, there exist holomorphic coordinates $\{z_1, z_2\}$ such that the holomorphic
(2, 0)-form on $N$ can be expressed as

$$\omega_J + i \omega_K = dz_1 \wedge dz_2.$$ 

Let $\kappa$ be the Kähler potential, i.e., we have

$$\omega_I = i \partial \bar{\partial} \kappa = i(\kappa_{11}dz_1 \wedge d\bar{z}_1 + \kappa_{12}dz_1 \wedge d\bar{z}_2 + \kappa_{21}dz_2 \wedge d\bar{z}_1 + \kappa_{22}dz_2 \wedge d\bar{z}_2),$$

where lower indices represent partial derivatives with respect to holomorphic or anti-holomorphic coordinates. As $N$ is hyperkähler hence Ricci-flat, we must have

$$\kappa_{11} \kappa_{22} - \kappa_{12} \kappa_{21} = \frac{1}{4},$$

For the convenience of calculation, it is useful to list the following equations

$$Idz_1 = idz_1, \quad Idz_2 = idz_2,$$

$$Jdz_1 = -2(\kappa_{21}d\bar{z}_1 + \kappa_{22}d\bar{z}_2),$$

$$Jdz_2 = 2(\kappa_{11}d\bar{z}_1 + \kappa_{12}d\bar{z}_2),$$

$$Kdz_1 = -2i(\kappa_{21}d\bar{z}_1 + \kappa_{22}d\bar{z}_2),$$

$$Kdz_2 = 2i(\kappa_{11}d\bar{z}_1 + \kappa_{12}d\bar{z}_2),$$

$$Id\bar{z}_1 = -id\bar{z}_1, \quad Id\bar{z}_2 = -id\bar{z}_2,$$

$$Jd\bar{z}_1 = -2(\kappa_{12}dz_1 + \kappa_{22}dz_2),$$

$$Jd\bar{z}_2 = 2(\kappa_{11}dz_1 + \kappa_{21}dz_2),$$

$$Kd\bar{z}_1 = 2i(\kappa_{12}dz_1 + \kappa_{22}dz_2),$$

$$Kd\bar{z}_2 = -2i(\kappa_{11}dz_1 + \kappa_{21}dz_2),$$

which tell us how $I$, $J$ and $K$ act on 1-forms on $N$.

By noticing that $X$ is naturally diffeomorphic to $\mathbb{C} \times N$, we may think of $z_i$ as functions locally defined on $X$, though they are no longer holomorphic. In the same
manner, we shall record how the complex structure \( J \) on \( X \) acts on 1-forms

\[
\begin{align*}
\mathfrak{J}d\zeta &= id\zeta, \\
\mathfrak{J}dz_1 &= i\alpha dz_1 - 2(\beta + i\gamma)(\kappa_{21}d\bar{z}_1 + \kappa_{22}d\bar{z}_2), \\
\mathfrak{J}dz_2 &= i\alpha dz_2 + 2(\beta + i\gamma)(\kappa_{11}d\bar{z}_1 + \kappa_{12}d\bar{z}_2), \\
\mathfrak{J}d\bar{\zeta} &= -id\bar{\zeta}, \\
\mathfrak{J}d\bar{z}_1 &= -i\alpha d\bar{z}_1 - 2(\beta - i\gamma)(\kappa_{12}dz_1 + \kappa_{22}dz_2), \\
\mathfrak{J}d\bar{z}_2 &= -i\alpha d\bar{z}_2 + 2(\beta - i\gamma)(\kappa_{11}dz_1 + \kappa_{21}dz_2).
\end{align*}
\]

Now consider a 1-form

\[
\theta = LD\zeta + Adz_1 + Bdz_2 + Cd\bar{z}_1 + Dd\bar{z}_2.
\]

If \( \theta \) is of type \((1,0)\), i.e., \( \mathfrak{J}\theta = i\theta \), then we have

\[
\begin{align*}
A &= \frac{2i\kappa_{12}}{\zeta}C - \frac{2i\kappa_{11}}{\zeta}D, \\
B &= \frac{2i\kappa_{22}}{\zeta}C - \frac{2i\kappa_{21}}{\zeta}D.
\end{align*}
\]

Let

\[
\theta_1 = 2i\frac{\kappa_{12}}{\zeta}dz_1 + 2i\frac{\kappa_{22}}{\zeta}dz_2 + d\bar{z}_1, \quad \theta_2 = 2i\frac{\kappa_{11}}{\zeta}dz_1 + 2i\frac{\kappa_{21}}{\zeta}dz_2 - d\bar{z}_2.
\]

It is easy to see that they are \((1,0)\)-forms. Moreover, we have

\[
\theta = LD\zeta + C\theta_1 - D\theta_2.
\]

**Lemma 4.3.1.**
\( \theta \) is a holomorphic \((1,0)\)-form if and only if

\[
2\zeta \bar{\partial}L = -C((1 - \alpha)\theta_1 - 2d\bar{z}_1) + D((1 - \alpha)\theta_2 + 2d\bar{z}_2)
\]
holds, where $C$ and $D$ satisfy

\begin{align}
\bar{\partial} C &= i(\beta - i\gamma)[C(\kappa_{112}\overline{\theta}_1 - \kappa_{212}\overline{\theta}_2) - D(\kappa_{111}\overline{\theta}_1 - \kappa_{211}\overline{\theta}_2)] \\
\bar{\partial} D &= i(\beta - i\gamma)[C(\kappa_{122}\overline{\theta}_1 - \kappa_{222}\overline{\theta}_2) - D(\kappa_{112}\overline{\theta}_1 - \kappa_{212}\overline{\theta}_2)].
\end{align}

(4.12)

**Proof.** The proof follows from straightforward calculation. It is quite clear that $\theta$ is holomorphic if and only if

\[\bar{\partial} L \wedge d\zeta + \bar{\partial} C \wedge \theta_1 - \bar{\partial} D \wedge \theta_2 + C\bar{\partial} \theta_1 - D\bar{\partial} \theta_2 = 0.\]

One can compute directly that

\[\partial \theta_1 = -\frac{1 + \alpha}{2\zeta} d\zeta \wedge \theta_1,\]
\[\bar{\partial} \theta_1 = -\frac{d\zeta}{\zeta} \wedge \left(\frac{1 - \alpha}{2}\theta_1 - d\overline{z}_1\right) - \frac{2i}{\zeta}(\kappa_{112}dz_1 \wedge d\overline{z}_1 + \kappa_{122}dz_1 \wedge d\overline{z}_2 + \kappa_{212}dz_2 \wedge d\overline{z}_1 + \kappa_{222}dz_2 \wedge d\overline{z}_2),\]
\[\partial \theta_2 = -\frac{1 + \alpha}{2\zeta} d\zeta \wedge \theta_2,\]
\[\bar{\partial} \theta_2 = -\frac{d\zeta}{\zeta} \wedge \left(\frac{1 - \alpha}{2}\theta_2 + d\overline{z}_2\right) - \frac{2i}{\zeta}(\kappa_{112}dz_1 \wedge d\overline{z}_1 + \kappa_{112}dz_1 \wedge d\overline{z}_2 + \kappa_{212}dz_2 \wedge d\overline{z}_1 + \kappa_{212}dz_2 \wedge d\overline{z}_2).\]

Therefore the holomorphicity of $\theta$ is equivalent to (4.11) and

\begin{align}
\frac{\zeta}{2i} (\bar{\partial} C \wedge \theta_1 - \bar{\partial} D \wedge \theta_2) &= C(\kappa_{112}dz_1 \wedge d\overline{z}_1 + \kappa_{122}dz_1 \wedge d\overline{z}_2 + \kappa_{212}dz_2 \wedge d\overline{z}_1 + \kappa_{222}dz_2 \wedge d\overline{z}_2) \\
&\quad - D(\kappa_{111}dz_1 \wedge d\overline{z}_1 + \kappa_{112}dz_1 \wedge d\overline{z}_2 + \kappa_{211}dz_2 \wedge d\overline{z}_1 + \kappa_{212}dz_2 \wedge d\overline{z}_2).
\end{align}

(4.13)

Observe that Equation (4.11) is (locally) solvable if and only if

\[\bar{\partial} C \wedge ((1 - \alpha)\theta_1 - 2d\overline{z}_1) = \bar{\partial} D \wedge ((1 - \alpha)\theta_2 + 2d\overline{z}_2).\]
Or in other words, there exists functions \( P, Q \) and \( R \) such that

\[
\bar{\partial} C = P((1 - \alpha)\theta_1 - 2d\bar{z}_1) + Q((1 - \alpha)\theta_2 + 2d\bar{z}_2), \\
\bar{\partial} D = -Q((1 - \alpha)\theta_1 - 2d\bar{z}_1) + R((1 - \alpha)\theta_2 + 2d\bar{z}_2).
\]

Plugging them in Equation (4.13) we get

\[
2P\kappa_{12} + 2Q\kappa_{11} = C\kappa_{112} - D\kappa_{111}, \\
2R\kappa_{11} - 2Q\kappa_{12} = C\kappa_{122} - D\kappa_{112}, \\
2P\kappa_{22} + 2Q\kappa_{21} = C\kappa_{212} - D\kappa_{211}, \\
2R\kappa_{21} - 2Q\kappa_{22} = C\kappa_{222} - D\kappa_{212}.
\]

These equations can be solved explicitly

\[
P = 2C(\kappa_{11}\kappa_{212} - \kappa_{21}\kappa_{112}) + 2D(\kappa_{11}\kappa_{111} - \kappa_{11}\kappa_{211}), \\
Q = 2C(\kappa_{22}\kappa_{112} - \kappa_{12}\kappa_{212}) + 2D(\kappa_{12}\kappa_{211} - \kappa_{22}\kappa_{111}), \\
= 2C(\kappa_{21}\kappa_{122} - \kappa_{11}\kappa_{222}) + 2D(\kappa_{11}\kappa_{212} - \kappa_{21}\kappa_{112}), \\
R = 2C(\kappa_{22}\kappa_{122} - \kappa_{12}\kappa_{222}) + 2D(\kappa_{12}\kappa_{212} - \kappa_{22}\kappa_{112}).
\]

Substituting \( P, Q \) and \( R \) above back into (4.13), we get exactly (4.12) and prove the lemma.

Now we are ready to compute the curvature term \( \text{Tr}(R \wedge R) \) with respect to the metric (4.8). For the simplicity of notations, we shall let \( s = 1 + |\zeta|^2 \) and use \( K = (\kappa^{jk}) \) to denote the inverse matrix of \( (\kappa_{jk}) \). In addition, \( \langle \cdot, \cdot \rangle \) is to be understood as the Hermitian metric associated to \( \omega \).

Let \( L_1d\zeta + C_1\theta_1 - D_1\theta_2 \) and \( L_2d\zeta + C_2\theta_1 - D_2\theta_2 \) be locally defined linearly inde-
pendent holomorphic \((1, 0)\)-forms. It is not hard to compute that
\[
\langle d\zeta, d\zeta \rangle = \frac{s^2}{2e^{2g}}; \\
\langle \theta_1, \theta_1 \rangle = \frac{s^2}{e^{2h+g}} \left( \kappa^{11} + \frac{4}{|\zeta|^2} \kappa^{22} \right) = \frac{s^3}{|\zeta|^2 e^{2h+g}} \kappa^{11}, \\
\langle \theta_1, \theta_2 \rangle = -\frac{s^3}{|\zeta|^2 e^{2h+g}} \kappa^{12}, \\
\langle \theta_2, \theta_2 \rangle = \frac{s^3}{|\zeta|^2 e^{2h+g}} \kappa^{22}.
\]

If we consider the local frame
\[
\{ d\zeta, (\zeta L_1)d\zeta + C_1(\zeta \theta_1) + D_1(-\zeta \theta_2), (\zeta L_2)d\zeta + C_2(\zeta \theta_1) + D_2(-\zeta \theta_2) \},
\]
then the Gram matrix \( H \) is given by
\[
H = \frac{s^2}{2e^{2g}} \left( \begin{array}{c} 1 \\ L \end{array} \right) \left( \begin{array}{c} 1 \\ \bar{L}^T \end{array} \right) + \frac{s^3}{e^{2h+g}} \left( \begin{array}{cc} 0 & 0 \\ 0 & \Lambda K \bar{\Lambda}^T \end{array} \right),
\]
where
\[
L = \left( \begin{array}{c} \zeta L_1 \\ \zeta L_2 \end{array} \right)
\]
and
\[
\Lambda = \left( \begin{array}{cc} C_1 & D_1 \\ C_2 & D_2 \end{array} \right).
\]

Let \( U = \Lambda K \bar{\Lambda}^T \), notice that
\[
det H = \frac{s^8}{2e^{4g+4h}} \det U,
\]
so we can compute that
\[
H^{-1} = \frac{2e^{2g}}{s^2} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{e^{2h+g}}{s^3} \left( \begin{array}{c} -\bar{L}^T \\ I_2 \end{array} \right) U^{-1} \left( \begin{array}{cc} L & I_2 \end{array} \right),
\]

68
where $I_2$ is the $2 \times 2$ identity matrix. Write

$$A = \frac{s^2}{2e^{2g}} \quad \text{and} \quad B = \frac{s^3}{e^{2k+g}}.$$  

It follows that

$$\partial \bar{H} = \partial A \begin{pmatrix} 1 \\ \bar{L} \end{pmatrix} \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} + A \begin{pmatrix} 0 \\ \partial \bar{L} \end{pmatrix} \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} + A \begin{pmatrix} 1 \\ \bar{L} \end{pmatrix} \begin{pmatrix} 0 & \partial L^T \\ 0 & 0 \end{pmatrix}$$

$$+ \partial B \begin{pmatrix} 0 & 0 \\ 0 & \bar{U} \end{pmatrix} + B \begin{pmatrix} 0 & 0 \\ 0 & \partial \bar{U} \end{pmatrix},$$

$$\bar{H}^{-1} \partial \bar{H} = \partial \log A \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial L^T \\ 0 & 0 \end{pmatrix} + A \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1} \partial \bar{L} \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix}$$

$$+ \partial \log B \begin{pmatrix} 0 & -L^T \\ 0 & I_2 \end{pmatrix} + \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \begin{pmatrix} 0 & \bar{L}^T \\ 0 & \partial \bar{L} \end{pmatrix}.$$  

As a consequence,

$$R = \bar{\partial}(\bar{H}^{-1} \partial \bar{H})$$

$$= \bar{\partial} \partial \log A \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} - \partial \log A \begin{pmatrix} 0 & \partial L^T \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial \partial L^T \\ 0 & 0 \end{pmatrix}$$

$$+ \bar{\partial} \left( \frac{A}{B} \right) \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1} \partial \bar{L} \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} - \frac{A}{B} \begin{pmatrix} \partial L^T \\ 0 \end{pmatrix} \bar{U}^{-1} \partial \bar{L} \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix}$$

$$+ \frac{A}{B} \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1} \partial \bar{L} \begin{pmatrix} 1 & L^T \\ 0 & 0 \end{pmatrix} - \frac{A}{B} \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \bar{U}^{-1} \partial \bar{L} \begin{pmatrix} 0 & \partial L^T \\ 0 & 0 \end{pmatrix}$$

$$+ \bar{\partial} \partial \log B \begin{pmatrix} 0 & -L^T \\ 0 & I_2 \end{pmatrix} + \partial \log B \begin{pmatrix} 0 & \bar{L}^T \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \partial L^T \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{U}^{-1} \partial \bar{U} \end{pmatrix}$$

$$+ \begin{pmatrix} -L^T \\ I_2 \end{pmatrix} \begin{pmatrix} 0 & \bar{L}^T \\ 0 & \partial \bar{L} \end{pmatrix}.$$
A lengthy calculation shows that

\begin{equation}
\text{Tr}(R) = \partial \bar{\partial} \log A + 2\partial \bar{\partial} \log B + \text{Tr}(\bar{\partial}(\bar{U}^{-1} \partial U))
\end{equation}

and

\begin{equation}
\text{Tr}(R \wedge R) = 2\partial \bar{\partial} \left( \frac{A}{B} W \right) + 2(\partial \bar{\partial} \log B)^2
\end{equation}

where \( W = \partial L^T \bar{U}^{-1} \partial \bar{L} \) and we have used the fact that \( (\partial \bar{\partial} \log A)^2 = 0 \).

Recall from (4.9) that if \( ||\Omega||_\omega \) is a constant, or equivalently, \( e^{h+g} = s^2 \), then \( \text{Tr}(R) = 0 \). From this and (4.14) we can deduce that

\begin{equation}
\text{Tr}(\bar{\partial}(\bar{U}^{-1} \partial U)) = 0.
\end{equation}

From the (restricted) holomorphic twistor fibration \( p : X \to \mathbb{C} \), we have a short exact sequence of holomorphic vector bundles

\[ 0 \to p^*(T^*)^{1,0} \mathbb{C} \to (T^*)^{1,0} X \xrightarrow{\eta} E' = (T^*)^{1,0} X / p^*(T^*)^{1,0} \mathbb{C} \to 0. \]

Moreover, the matrix \( U \) is the Gram matrix of the holomorphic frame of the relatively cotangent bundle \( E' \) induced by

\[ \{ (\zeta L_1) d\zeta + C_1(\zeta \theta_1) + D_1(-\zeta \theta_2), (\zeta L_2) d\zeta + C_2(\zeta \theta_1) + D_2(-\zeta \theta_2) \} \]

with respect to the natural metric scaled by \( s \). Let \( F' \) be the curvature form of \( E' \) with respect to this metric, then we have

\begin{equation}
F' = \bar{\partial}(U^{-1} \partial U).
\end{equation}
By making use of (4.16) and (4.17), we can simplify (4.15) to

(4.18) \[ \text{Tr}(R \wedge R) = 2\bar{\partial}\bar{\partial} \left( \frac{A}{B} W \right) + 2(\bar{\partial}\partial \log B)^2 + \text{Tr}(F' \wedge F'). \]

The quantity $W$ can be computed explicitly using (4.11). Notice that

\[ 2\bar{\partial}L^T = \left( 2d\bar{z}_1 - (1 - \alpha)\theta_1, 2d\bar{z}_2 + (1 - \alpha)\theta_2 \right) \Lambda^T, \]

it follows that

\[
W = \frac{1}{4} \left( 2d\bar{z}_1 - (1 - \alpha)\theta_1, 2d\bar{z}_2 + (1 - \alpha)\theta_2 \right) K^{-1} \left( 2d\bar{z}_1 - (1 - \alpha)\bar{\theta}_1, 2d\bar{z}_2 + (1 - \alpha)\bar{\theta}_2 \right)
\]

\[
= \frac{2i}{s} (\alpha \omega_I + \beta \omega_J + \gamma \omega_K).
\]

Consequently, the anomaly cancellation equation (3.3) reduces to

(4.19) \[
i\bar{\partial}\bar{\partial} \left( \frac{e^{2h}}{s^2} \left( e^g - \frac{\alpha'}{2} e^{-g} \right) (\alpha \omega_I + \beta \omega_J + \gamma \omega_K) \right)
\]

\[
= \frac{\alpha'}{4} \left[ 2(\bar{\partial}\partial \log B)^2 + \text{Tr}(F' \wedge F') - \text{Tr}(F \wedge F) \right]
\]

and we are free to choose functions $g$ and $h$.

The simplest way to let (4.19) hold is to set $g = \frac{1}{2} \log \frac{\alpha'}{2}$, $F = F'$ and choose appropriate $h$ such that

(4.20) \[(\bar{\partial}\partial \log B)^2 = 0.\]

Recall that $B = s^3/e^{2h+g}$, hence (4.20) holds trivially if $h$ is a constant say $h \equiv 0$. In this case, the metric (4.8) is conformal to the product of hyperkähler metric on $N$ and the Euclidean metric on $\mathbb{C}$, hence conformally Ricci-flat. It should be pointed out that this metric is not complete, however this does not raise any problem for the use of gluing. Intuitively we can think of $\omega$ as a metric on the singular space $\mathbb{C}\mathbb{P}^1 \times N/\{\infty\} \times N$. Therefore we have demonstrated that the anomaly cancellation
equation (3.3) can be solved by choosing both $g$ and $h$ to be constant and $E$ to be the relative cotangent bundle of $p : X \to \mathbb{C}$.

It is also possible that (4.20) holds for nonconstant $h$. To find such $h$, one usually needs to know the explicit hyperkähler metric on $N$. We will give an example later in this section.

In our last section, $N$ is taken to be flat $T^4$ hence $F' = 0$ and the Hermitian-Yang-Mills equation (3.2) holds automatically. In general, neither $N$ is flat nor $F'$ trivial. Therefore in order to solve the whole Strominger system, we need to prove

**Theorem 4.3.2.**

$F'$ solves the Hermitian-Yang-Mills equation (3.2), i.e.,

$$F' \wedge \omega^2 = 0$$

for arbitrary $g$ and $h$.

By the product structure $X = \mathbb{C} \times N$, we can decompose the space of 2-forms on $X$ as

$$\Omega^2(X) = \Omega^2(\mathbb{C}) \oplus \Omega^1(\mathbb{C}) \otimes \Omega^1(N) \oplus \Omega^2(N).$$

Moreover the complex structure $\mathcal{I}$ is compatible with this decomposition, so we have

$$\Omega^{1,1}(X) = \Omega^{1,1}(\mathbb{C}) \oplus \Omega^{1,0}(\mathbb{C}) \otimes \Omega^{0,1}(N) \oplus \Omega^{0,1}(\mathbb{C}) \otimes \Omega^{1,0}(N) \oplus \Omega^{1,1}(N).$$

We first prove the following lemma

**Lemma 4.3.3.**

Every entry of $F'$ is contained in the space

$$\Omega^{1,0}(\mathbb{C}) \otimes \Omega^{0,1}(N) \oplus \Omega^{0,1}(\mathbb{C}) \otimes \Omega^{1,0}(N) \oplus \Omega^{1,1}(N).$$

**Proof.** Recall that $U = \Lambda K \bar{\Lambda}^T$, so we have

$$F' = \bar{\partial}(U^{-1} \partial U) = \bar{\partial}((\Lambda^T)^{-1} K^{-1} \bar{\Lambda}^{-1} \partial \Lambda \cdot K \Lambda^T) + \bar{\partial}((\Lambda T)^{-1} K^{-1} \partial K \cdot \Lambda^T) + \bar{\partial}((\Lambda^T)^{-1} \partial \Lambda^T)).$$

72
Observe that (4.12) can be rewritten as

$$\bar{\partial} \Lambda = i(\beta - i\gamma) \Lambda \left( \begin{array}{cc}
\kappa_{112} \bar{\theta}_1 - \kappa_{212} \bar{\theta}_2 & \kappa_{122} \bar{\theta}_1 - \kappa_{222} \bar{\theta}_2 \\
\kappa_{211} \bar{\theta}_2 - \kappa_{111} \bar{\theta}_1 & \kappa_{221} \bar{\theta}_2 - \kappa_{121} \bar{\theta}_1
\end{array} \right),$$

which does not contain any component from $\Omega^1(\mathbb{C})$. Similarly $\bar{\partial} \bar{K}$ contains only components in $\Omega^{1,0}(N)$. The lemma follows directly from these two observations.

Now we proceed to prove Theorem 4.3.2.

**Proof.** From (4.10), we see that $\omega^2$ lives in the space

$$\Omega^2(\mathbb{C}) \otimes \Omega^2(N) \oplus \Omega^4(N).$$

Therefore the only component of $F'$ that would contribute in $F' \wedge \omega^2$ is its $\Omega^{1,1}(N)$-part. The key point is that the $\Omega^{1,1}(N)$-part of $F'$ can be computed fiberwise.

Fix $\zeta \in \mathbb{C}$ and let $N_\zeta$ be the fiber of the holomorphic twistor fibration over $\zeta$. Notice that $(N_\zeta, \omega_\zeta := \alpha \omega_I + \beta \omega_J + \gamma \omega_K)$ is hyperkähler. Moreover, $E'|_{N_\zeta}$ is the cotangent bundle of $N_\zeta$ with its hyperkähler metric. It is a well-known fact that hyperkähler 4-manifolds are anti-self-dual, thus

$$F'_{\Omega^{1,1}(N)} \wedge \omega_\zeta = 0.$$

The theorem follows directly.

In summary, the main theorem we have proved is the following:

**Theorem A.**

Let $N$ be a hyperkähler 4-manifold and let $p : Z \to \mathbb{C}P^1$ be its holomorphic twistor fibration. By removing an arbitrary fiber of $p$ from $Z$, we get a noncompact 3-fold $X$ which has trivial canonical bundle. It is worth mentioning that the holomorphic structure of $X$ depends on the choice of fiber. For such $X$'s, we can always construct explicit solutions to the Strominger system on them. These spaces include $\mathbb{C}^3$ and the resolved conifold $\mathcal{O}(-1, -1)$ as special examples.
Now let us take a closer look at the solutions constructed above for $N = \mathbb{R}^4$.

Identify $\mathbb{R}^4$ with the space of quaternions $\mathbb{H}$, then left multiplication by $i, j$ and $k$ defines the standard hyperkähler structure on $\mathbb{R}^4$. We can construct the space $X$ by removing the fiber with complex structure $-I$ at infinity. Actually, $X$ is biholomorphic to $\mathbb{C}^3$. An explicit biholomorphism $\psi : \mathbb{C}^3 = \mathbb{C} \times \mathbb{C}^2 \to X = \mathbb{C} \times \mathbb{R}^4$ can be written down as

$$(\zeta, u_1, u_2) = \psi(\zeta, w_1, w_2) = \left( \zeta, \frac{w_1 - i\zeta \bar{w}_2}{1 + |\zeta|^2}, \frac{w_2 + i\zeta \bar{w}_1}{1 + |\zeta|^2} \right).$$

Or conversely,

$$\begin{cases}
  w_1 = u_1 + i\zeta \bar{u}_2 \\
  w_2 = u_2 - i\zeta \bar{u}_1,
\end{cases}$$

where $u_1 = x^1 + ix^2, u_2 = x^3 + ix^4$ and $u_1 + u_2 j = x^1 + x^2 i + x^3 j + x^4 k$ parameterizes $\mathbb{H} = \mathbb{R}^4$. In terms of the holomorphic coordinates $\{\zeta, w_1, w_2\}$ on $\mathbb{C}^3$, the 2-form $\alpha \omega_I + \beta \omega_J + \gamma \omega_K$ can be expressed explicitly as

$$(4.21) \quad \frac{i}{2s} (dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2 + (|u_1|^2 + |u_2|^2) d\zeta \wedge d\bar{\zeta} + iu_2 dw_1 \wedge d\bar{\zeta} - iu_1 dw_2 \wedge d\bar{\zeta} - i\bar{u}_2 d\zeta \wedge d\bar{w}_1 + i\bar{u}_1 d\zeta \wedge d\bar{w}_2).$$

Now let us consider Equation (4.20)

$$(\bar{\partial} \partial \log B)^2 = 0.$$
For simplicity, let us assume that \( h : \mathbb{R}^4 \to \mathbb{R} \) is a radial function, i.e., \( h = h(\rho) \)
where
\[
\rho = |u_1|^2 + |u_2|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.
\]
Therefore
\[
dh = h' d\rho
\]
and
\[
2i\bar{\partial} \partial h = d\bar{\partial} dh
\]
\[
= h'' d\rho \wedge (\alpha I + \beta J + \gamma K) d\rho + h'(d\alpha \wedge Id\rho + d\beta \wedge Jd\rho + d\gamma \wedge Kd\rho)
\]
\[
- 4h'(\alpha \omega_I + \beta \omega_J + \gamma \omega_K).
\]
One can verify that
\[
4(\bar{\partial} \partial h)^2
\]
\[
= 2(h')^2 (\alpha Jd\rho \wedge Kd\rho + \beta Kd\rho \wedge Id\rho + \gamma Id\rho \wedge Jd\rho) \wedge \omega_{\mathbb{CP}^1} - 32h'(ph')' \text{vol}_{\mathbb{R}^4}
\]
\[
- 8h'(ph')' ((\alpha d\beta - \beta d\alpha) \wedge * Kd\rho + (\beta d\gamma - \gamma d\beta) \wedge * Id\rho + (\gamma d\alpha - \alpha d\gamma) \wedge * Jd\rho),
\]
where * is the standard Hodge star operator on \( \mathbb{R}^4 \). In addition we have
\[
4\bar{\partial} \partial h \wedge (3\bar{\partial} \partial \log s - \bar{\partial} \partial g)
\]
\[
= (h'' d\rho \wedge (\alpha I + \beta J + \gamma K) d\rho - 4h'(\alpha \omega_I + \beta \omega_J + \gamma \omega_K)) \wedge (2i\bar{\partial} \partial \log g + 3\omega_{\mathbb{CP}^1}).
\]
By using the identity
\[
d\rho \wedge Id\rho + 4\rho \cdot \omega_I = Jd\rho \wedge Kd\rho,
\]
and assuming \( g \) is a constant, it follows that (4.22) holds if and only if \( h' = 0 \) or
\[
h' = -\frac{3}{2\rho}.
\]
In both cases we solve the Strominger system.
(a). The case \( h' = 0 \).

The metric on \( X = \mathbb{C} \times \mathbb{R}^4 \) is essentially of the form
\[
\omega = \frac{1}{s^2} \left( \alpha \omega_I + \beta \omega_J + \gamma \omega_K + \frac{i}{2} d\zeta \wedge d\bar{\zeta} \right).
\]
which is conformal to the Euclidean metric on $X$. However this metric has rather complicated expression in terms of standard complex coordinates $(\zeta, w_1, w_2)$ on $\mathbb{C}^3$:

$$\omega = \frac{2i}{(1 + |\zeta|^2)^2}d\zeta \wedge d\overline{\zeta} + \frac{i}{2(1 + |\zeta|^2)^3}(dw_1 \wedge d\overline{w}_1 + dw_2 \wedge d\overline{w}_2$$

$$+ iu_2dw_1 \wedge d\overline{\zeta} - iw_1dw_2 \wedge d\overline{\zeta} - i\overline{u}_2d\zeta \wedge d\overline{w}_1 + i\overline{u}_1d\zeta \wedge d\overline{w}_2$$

$$+ (|u_1|^2 + |u_2|^2)d\zeta \wedge d\overline{\zeta},$$

where

$$(u_1, u_2) = \left(\frac{w_1 - i\zeta\overline{w}_2}{1 + |\zeta|^2}, \frac{w_2 + i\zeta\overline{w}_1}{1 + |\zeta|^2}\right).$$

Direct calculation shows that though its sectional curvature is not bounded from below, however it is bounded from above by a positive constant. Same conclusion applies to Ricci and scalar curvatures as well.

(b). The case $h' = \frac{-3}{2\rho}$.

In this case $e^h \sim \rho^{-3/2}$, so the metric $\omega$ is only defined on $\mathbb{C} \times (\mathbb{R}^4 \setminus \{0\}) \cong \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\})$. On each copy of $\mathbb{R}^4 \setminus \{0\}$, the restricted metric is conformally flat, with non-positive sectional curvature. The curvature properties of $\mathbb{C} \times (\mathbb{R}^4 \setminus \{0\})$ behave in the same way as what we have seen in the $h' = 0$ case.
Appendix A

On Chern-Ricci-Flat Balanced Metrics

Let $X$ be a Kähler manifold with trivial canonical bundle. A fundamental question is whether $M$ admits a (complete) Ricci-flat Kähler metric or not. For $X$ compact, this was answered affirmatively by Yau’s famous solution to the Calabi conjecture [116, 118]. However, when $X$ is noncompact, the same problem is far from being solved.

For instance, let $M$ be any Kähler manifold, then we know that the total space of the canonical bundle of $M$, denoted by $K_M$, is Kähler and has itself trivial canonical bundle. Therefore we may ask when does $K_M$ support a Kähler Ricci-flat metric. This problem was first studied by Calabi [20], where he showed that if $M$ admits a Kähler-Einstein metric, then one can write down a Kähler Ricci-flat metric on $K_M$. A relative recent progress in this direction was made by Futaki [60], where he proved that such a metric exists if $M$ is toric Fano.

The Calabi ansatz can be rephrased as follows. Let $(M, \omega)$ be a Kähler manifold. By choosing a set of holomorphic coordinates $\{z^1, \ldots, z^n\}$ on $M$, we can trivialize $K_M$ by $dz^1 \wedge \cdots \wedge dz^n$ locally. Let $t$ parameterizes the fiber of $K_M$ under this trivialization, then $\{z^1, \ldots, z^n, t\}$ forms a set of holomorphic coordinates on $K_M$ and

$$\Omega = dz^1 \wedge \cdots \wedge dz^n \wedge dt$$
is a globally defined nowhere vanishing holomorphic volume form.

In terms of coordinates \( \{z^1, \ldots, z^n\} \), the Kähler form \( \omega \) can be written as

\[
\omega = i h_{jk} dz^j \wedge dz^k.
\]

It follows that \( h = \det(h_{jk}) \) is a positive function. Notice that the Kähler metric on \( M \) naturally induces an Hermitian metric on \( K_M \), which can be expressed as

\[
\omega_0 = \omega + \frac{i}{h} (dt - t\partial \log h) \wedge (d\bar{t} - \bar{t}\partial \log h).
\]

Let \( R : K_M \to \mathbb{R} \) be the norm square function of fibers of \( K_M \to M \). Clearly \( R = \frac{|t|^2}{h} \)
and the metric \( \omega_0 \) has the form

\[
\omega_0 = \omega + i \frac{\partial R \wedge \bar{\partial} R}{R}.
\]

In general, \( \omega_0 \) is not a Kähler metric. It turns out that \( \omega_0 \) is Kähler if and only if

\[
\partial \bar{\partial} \log R = -\partial \bar{\partial} \log h = 0,
\]

i.e., \( (M, \omega) \) is Kähler Ricci-flat.

To get a better behaved metric, we can modify \( \omega_0 \) by some conformal factors. Let \( u, v \) be smooth functions on \( M \), and \( f, g \) be smooth functions of \( R \). Then one can cook up a new Hermitian metric

\[
\omega_{u,v,f,g} = e^{u+f} \omega + i e^{v+g} \frac{\partial R \wedge \bar{\partial} R}{R}.
\]

It is not hard to check that \( \Omega \) is of constant length if and only if

\[
(A.1) \quad v = -nu \quad \text{and} \quad g = -nf + c
\]

for some constant \( c \). Assuming this, if we further want the metric \( \omega_{u,v,f,g} \) to be Kähler, then \( u \) must be a constant. Without loss of generality we may assume that \( u = 0 \),

78
and we still need
\[ e^f \partial f \wedge \omega - ie^{-n f + c} \partial R \wedge \bar{\partial} \log R = 0. \]
In other words,
\[ e^{(n+1) f - c} f' \omega = i \partial \bar{\partial} \log R = -i \partial \bar{\partial} \log h = Ric(\omega). \]
We see immediately that this equation has a solution if and only if \( \omega \) is Kähler-Einstein, in which case we get the Calabi ansatz.

In this appendix, we shall consider the case that \( \omega_{u,v,f,g} \) is balanced and Chern-Ricci-flat, i.e.,
\[ d(\omega_{u,v,f,g}^n) = 0 \quad \text{and} \quad \|\Omega\|_{\omega_{u,v,f,g}} \equiv \text{constant}. \]
As we have seen, such metric appears naturally in compactification of heterotic superstrings in the metric product model (cf. Model 2 in Section 3.1). Geometrically this condition is interpreted as that \( \Omega \) is parallel under the Bismut-Strominger connection.

Now let us impose the balanced Chern-Ricci-flat condition on \( \omega_{u,v,f,g} \). For the “Chern-Ricci-flat” part, i.e., \( \|\Omega\|_{\omega_{u,v,f,g}} \equiv \text{constant} \), we still need (A.1) as before. Plugging this in and we can compute that
\[ \omega_{u,v,f,g}^n = e^{n(u+f)} \omega^n + i e^{c - u - f} \omega^{n-1} \wedge \frac{\partial R \wedge \bar{\partial} R}{R}. \]
Hence
\[ \partial(\omega_{u,v,f,g}^n) = e^{n(u+f)} \partial f \wedge \omega^n - i e^{c - u - f} \omega^{n-1} \wedge \partial R \wedge \bar{\partial} \log R - i e^{c - u - f} \omega^{n-1} \wedge \partial u \wedge \frac{\partial R \wedge \bar{\partial} R}{R}. \]
As there are no other terms to cancel the last term in the above equation, so we need \( \partial u = 0 \) to make \( \omega_{u,v,f,g} \) balanced. By choosing \( u = 0 \), the balancing condition is reduced to
\[ e^{nf} \partial f \wedge \omega^n = i e^{c-f} \omega^{n-1} \wedge \partial R \wedge \bar{\partial} \log R, \]
or equivalently,

\[ e^{(n+1)f - c f'} \omega^n = -i n \omega^{n-1} \partial \bar{\partial} \log h = s \cdot \omega^n, \]

where \( s \) is the scalar curvature function of \( M \) up to a positive constant. From the calculation we conclude that this is possible if and only if \((M, \omega)\) has constant scalar curvature.

Constant scalar curvature Kähler metrics (cscK) has been studied extensively as a special case of extremal Kähler metrics [21]. It is believed that the existence of cscK metrics is equivalent to certain stability condition in the sense of algebraic geometry.

Notice that in the derivation of Chern-Ricci-flat balanced metric on \( K_M \), what we actually need is that \( \omega \) is balanced instead of being Kähler. In such a case, \( s \) is known as the Chern scalar curvature, which is in general different from the scalar curvature in Riemannian geometry. Thus we have proved the following generalization of Calabi’s result.

**Theorem A.0.4.**

If \( M \) admits a balanced metric with constant Chern scalar curvature, then \( K_M \) admits a Chern-Ricci-flat balanced metric.

Recall that on a complex \( n \)-fold \( X \), a balanced metric \( \omega \) defines the so-called balanced class

\[
\left[ \frac{\omega^{n-1}}{(n-1)!} \right] \in H^{n-1, n-1}_{BC}(X) = \frac{\text{d-closed \((n-1, n-1)\)-forms}}{\partial \bar{\partial} \text{-exact \((n-1, n-1)\)-forms}}.
\]

The balanced version of Calabi (Gauduchon) conjecture [107] in general case is still open. In particular, for balanced manifolds with trivial canonical bundle, this conjecture implies the existence of Chern-Ricci-flat balanced metrics in any given balanced class. We refer to [105] for recent progresses on this problem.

Theorem A.0.4 implies that Chern-Ricci-flat balanced metrics should be viewed as a balanced analogue of extremal Kähler metrics. We shall justify this claim by the following consideration.

On a compact balanced manifold \((X, \omega)\), a very useful property is that the total
Chern scalar curvature
\[ \int_X s \cdot \frac{\omega^n}{n!} = \int_X \rho \wedge \frac{\omega^{n-1}}{(n-1)!} \]
depends only on the complex structure of \( X \) and the balanced class. Here \( \rho = -i\partial \bar{\partial} \log h \) is the Chern-Ricci form. As an analogue of Kähler case, it is natural to consider the variational problem associated to the Calabi-type functional [21]
\[ S(\omega) = \int_X s^2 \cdot \frac{\omega^n}{n!}, \]
where \( \frac{\omega^{n-1}}{(n-1)!} \) varies in the given balanced class. Since we can modify \( \omega \) by \( i\partial \bar{\partial} \alpha \), where \( \alpha \) is any \((n-2, n-2)\)-form, we expect the associated Euler-Lagrange equation is an equation of \((n-2, n-2)\)-forms, or dually, a \((2, 2)\)-form equation.

Indeed, it is not hard to derive the following

**Theorem A.0.5.**

A balanced metric \( \omega \) is a critical point of \( S(\omega) \) if and only if it satisfies

\[ 2(n-1)i\partial \bar{\partial} s \wedge \rho = i\partial \bar{\partial}((2\Delta s + s^2)\omega), \]

where \( \Delta_c \) is the complex Laplacian defined by
\[ \Delta f = \Lambda(i\partial \bar{\partial} f) = h^{k\bar{j}} \frac{\partial^2 f}{\partial z^{\bar{j}} \partial z^k}. \]

As an analogue of Kähler case, we shall call such balanced metrics *extremal*.

From Equation (A.2), we have the following observations:

(a). If the background metric is non-Kähler, one can easily show that \( i\partial \bar{\partial} \omega \neq 0 \). Hence \( s = \text{constant} \) is an extremal balanced metric if and only if \( s = 0 \). In particular, Chern-Ricci-flat balanced metrics are extremal. An intuitive reason is that there is no analogue of Kähler-Einstein metrics with nonzero Einstein constant on non-Kähler balanced manifolds. Indeed, if there is a smooth function \( f \) such that
\[ \rho = f \cdot \omega, \]
one can deduce that $\rho \equiv 0$.

(b). If there exists a $(1,0)$-form $\mu$ such that

\[(A.3) \quad 2(n - 1)s \cdot \rho = (2\Delta s + s^2)\omega + \bar{\partial}\mu + \partial\bar{\mu},\]

then (A.2) holds. On the other hand, if (A.2) holds, then condition (A.3) is automatically satisfied if $H_{A}^{1,1}(X) = 0$, where $H_{A}^{*}(X)$ is the Aeppli cohomology group of $X$. In this case, by taking trace of (A.3), we get

\[2(n - 1)s^2 = 2n\Delta s + ns^2 + \Lambda(\bar{\partial}\mu + \partial\bar{\mu}).\]

Since the last term is of divergence form, by integration over $X$, we get

\[(n - 2) \int_{X} s^2 \cdot \omega^n \frac{n!}{n!} = 0.\]

As we always assume that $n > 2$ (otherwise $\omega$ is Kähler), we conclude that $s \equiv 0$.

(c). Assuming $s \equiv 0$, if we further assume that $0 = c_1(X) \in H_{BC}^{1,1}(X)$, then $\rho = i\partial\bar{\partial}f$ for some globally defined real function $f$. Therefore

\[0 = s = \Lambda\rho = \Delta f.\]

Hence by maximal principle $f$ is a constant and $\rho \equiv 0$.

A very important class of non-Kähler Calabi-Yau 3-folds is of the form $X_k = \#_k(S^3 \times S^3)$ for $k \geq 2$ [50, 94], which can be constructed from projective Calabi-Yau 3-folds by taking conifold transitions (cf. Section 2.4). Moreover, these manifolds satisfy the $\partial\bar{\partial}$-lemma and admit balanced metrics [53]. Therefore conditions in (b) and (c) above are satisfied. Consequently we have

**Corollary A.0.6.**

Extremal balanced metrics on $X_k$ are exactly those Chern-Ricci-flat balanced metrics.
Hopefully this point of view will be useful in proving the balanced version of Calabi (Gauduchon) conjecture for $X_k$’s.


86


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87


88


