**PROBLEM SET 5**

**Problem 1.** Recall that for any pointed space $X$, the loop space $\Omega X$ is an $H$-space. Deduce that $H_*(\Omega X)$ is a ring. Sketch an argument why the homology Serre spectral sequence for $\Omega X$ has a multiplicative structure of the form $E^r_{p',q'} \rightarrow E^{r}_{p',q'+q'}$. Use all this to show that $H_*(\Omega S^n, \mathbb{Z}) \simeq \mathbb{Z} [x]$, where $x$ is a generator in $n - 1$.

**Problem 2.** Let $n \geq 2$ and $d \neq 0$. Let $f_d : S^n \rightarrow S^n$ be of degree $d$, and write $F_d$ for the homotopy fiber of $f_d$.

1. Compute the homology of $F_d$.
2. Determine the structure of the homology spectral sequence for the fiber sequence $\Omega S^n \rightarrow F_d \rightarrow S^n$, ignoring signs.

**Problem 3.** Compute $H^*(K(\mathbb{Z}, 2))$ from the Serre spectral sequence for $S^1 \rightarrow * \rightarrow K(\mathbb{Z}, 2)$ (i.e. without using that $K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty$).

**Problem 4.** Let $n$ be even. Show that $H^*(\Omega S^{n+1}, \mathbb{Z})$ is given by a divided power algebra, i.e.

$$H^*(\Omega S^{n+1}, \mathbb{Z}) \simeq \mathbb{Z} [\gamma_k | k \geq 1] / (\gamma_k^k = k! \gamma_k),$$

where $\gamma_k$ has degree $kn$.

**Problem 5.** Let $\xi$ be a fibration with fiber $S^{n-1}$. Show that if $n$ is odd then $e(\xi)$ has order 2. Given an example (for odd $n$) such that $e(\xi) \neq 0$.

**Problem 6.** Let $\xi$ be an $R$-oriented $n$-plane bundle on $X$. Show that the image of the Thom class under the restriction $H^n(Th(\xi)) \rightarrow H^n(\xi \setminus 0) \rightarrow H^n(X)$ coincides (up to a unit) with the Euler class of the associated spherical bundle.