PROBLEM SET 5

Problem 1. Recall that for any pointed space X, the loop space ΩX is an H-space. Deduce that $H_*(\Omega X)$ is a ring. Sketch an argument why the homology Serre spectral sequence for ΩX has a multiplicative structure of the form $H_q(\Omega X) \otimes E^r_{p',q'} \to E^r_{p',q+q'}$.

Use all this to show that $H_*(\Omega S^n, \mathbb{Z}) \simeq \mathbb{Z}[x]$, where x is a generator in $n-1$.

Problem 2. Let $n \geq 2$ and $d \neq 0$. Let $f_d : S^n \to S^n$ be of degree d, and write F_d for the homotopy fiber of f_d.

(1) Compute the homology of F_d.
(2) Determine the structure of the homology spectral sequence for the fiber sequence $\Omega S^n \to F_d \to S^n$, ignoring signs.

Problem 3. Compute $H^*(K(\mathbb{Z},2))$ from the Serre spectral sequence for $S^1 \to * \to K(\mathbb{Z},2)$ (i.e. without using that $K(\mathbb{Z},2) \simeq \mathbb{CP}^\infty$).

Problem 4. Let n be even. Show that $H^*(\Omega S^{n+1}, \mathbb{Z})$ is given by a divided power algebra, i.e.

$$H^*(\Omega S^{n+1}, \mathbb{Z}) \simeq \mathbb{Z}[\gamma_k | k \geq 1]/(\gamma_1^k = k!\gamma_k),$$

where γ_k has degree kn.

Problem 5. Let ξ be a fibration with fiber S^{n-1}. Show that if n is odd then $e(\xi)$ has order 2. Given an example (for odd n) such that $e(\xi) \neq 0$.

Problem 6. Let ξ be an R-oriented n-plane bundle on X. Show that the image of the Thom class under the restriction $H^n(Th(\xi)) \to H^n(\xi) \to H^n(X)$ coincides (up to a unit) with the Euler class of the associated spherical bundle.