## PROBLEM SET 4

*Problem* 1. Compute all pages and differentials of the (homological) Leray-Serre spectral sequence for the Hopf fibration $S^3 \to S^2$.

*Problem* 2. Let $p : X \to B$ be a fibration with fiber $F$, with $B$ and $F$ CW complexes.

(1) Suppose $F$ is connected. Show that the composite
$$H_s(X) \to H_s(X)/F_{s-1}H_s(X) \simeq E^\infty_{s,0} \hookrightarrow E^2_{s,0} \simeq H_s(B)$$
is $p_* : H_s(X) \to H_s(B)$.

(2) Suppose that $B$ is simply connected. Show that the composite
$$H_s(F) \simeq E^2_{0,s} \to E^\infty_{0,s} \simeq F_0H_s(X) \to H_s(X)$$
is $i_* : H_s(F) \to H_s(X)$.

*Problem* 3. Given an example of a filtered complex $F_*C$ such that $\cap_s F_sC = 0$ but $\cap_s F_sH_*C \neq 0$.

*Problem* 4. Let $F_*C$ be a filtered complex. Define a new filtration by $\tilde{F}_pC_n = F_{p-n}C_n$. Check that this is a filtration, and proved a relationship between the associated spectral sequences.

*Problem* 5 (double credits). We put $\mathbb{Z}/p^\infty = \mathrm{colim}_n\, \mathbb{Z}/p^n$. Given an abelian group $A$, define a filtration by $F_{-1}A = 0$ and for $s \geq 0$
$$F_sA = ker(p^{s+1} : A \to A).$$

Let $C$ be a chain complex of free abelian groups such that each group $H_n(C)$ is finitely generated. Give $C\otimes\mathbb{Z}/p^\infty$ the filtration arising from the above construction.

(1) Compute $(E^0, d^0)$ and $(E^1, d^1)$ of the associated spectral sequence. How is $d^1$ related to the short exact sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$?
(2) Sketch the spectral sequence in the case $C_i = \mathbb{Z}$ for $i = 0, 1$, $C_i = 0$ else, and $d : C_1 \to C_0$ given by multiplication by $p^n$.
(3) Multiplication by $p$ defines an operator $F_sC \to F_{s-1}C$. Describe its effect on the spectral sequence.
(4) Use this operator to "solve the extension problems" and describe $H_*(C \otimes \mathbb{Z}/p^\infty)$ in terms of this spectral sequence.
(5) Deduce that if $C \to C'$ is a quasi-isomorphism (same assumptions on $C'$ as on $C$), then so is $C \otimes \mathbb{Z}/p^\infty \to C' \otimes \mathbb{Z}/p^\infty$.
(6) Explain what this information says about $H_*(C)$ itself.
(7) There is a short exact sequence $0 \to C \xrightarrow{p} C \to C/p \to 0$, inducing an exact couple $H_*(C) \xrightarrow{p} H_*(C) \to H_*(C/p) \to \dots$. Explain how the spectral sequence associated to this exact couple relates to the spectral sequence arising from above filtration.