PROBLEM SET 1

Problem 1. Show that any (small) limit can be written as an equalizer between two products.

Here an equalizer means a limit of a diagram of the form $A \rightrightarrows B$ and a product means a limit of a discrete diagram, i.e. the indexing category has only identity morphisms.

Problem 2. Suppose that $F, F' : C \to D$ are both left adjoint to $G : D \to C$. Construct a natural isomorphism $F \cong F'$ and discuss its uniqueness.

Problem 3. Let $I$ be a small category and $C$ any category such that all $I$-indexed colimits in $C$ admit colimits. Denote by $\Delta : C \to \text{Fun}(I, C)$ the “constant diagram” functor, sending $c \in C$ to the diagram $I \ni i \mapsto c, \alpha : i \to j \mapsto \text{id}_c$.

Construct a left adjoint of $C$.

Problem 4. Let $S^\infty = \text{colim}_i S^i$, where the transition maps come from the evident inclusions $\mathbb{R}^i \to \mathbb{R}^{i+1}$.

Show that $S^\infty$ is contractible.

Problem 5. Let $i : A \rightarrow B$, $j : A' \rightarrow B'$ be CW pairs, where $A, A'$ are CW complexes. Show that $i \times j : A \times A' \hookrightarrow B \times B'$ is a CW pair.

Problem 6. Consider the “quasi-circle” $X$ obtained by taking the graph of $\sin(x^{-1})$ on $(0,1)$ and adding a path connecting $(0,0)$ to $(1,0)$ without touching the graph. Compute the homotopy groups of $X$. Deduce that $X$ is weakly contractible. Is $X$ contractible?