To Lily, for not being by my side.

Contents

Preface

Alexandre Grothendieck conceived his definition of *motives* in the 1960s. By that time, it was already established that there exist several cohomology theories for, say, smooth projective algebraic varieties defined over a given field k, and A. Weil's brilliant insight about counting points over finite fields via Lefschetz trace formula was validated.

With his characteristic passion for unification and "naturality", Grothendieck wanted to construct a universal cohomology theory (with, say, coefficients R) that had to be a functor h from the category $\mathrm{Var}(k)$ of smooth k-varieties to an abelian tensor category $\mathrm{Mot}(k)$ of "(pure) motives" (or $\mathrm{Mot}(k)_R$, where R is a ring of coefficients), satisfying a minimal list of expected properties.

Grothendieck also suggested a definition of $\mathrm{Mot}(k)$ and of the motivic functor. It consisted of several steps.

At the first step, one keeps objects of $\mathrm{Var}(k)$, but replaces its morphisms by correspondences. This passage implies that morphisms $Y \to X$ now form an *additive group*, or even a R-module rather than simply a set. Moreover, correspondences themselves are not just cycles on $X \times Y$ but *classes* of such cycles modulo an "adequate" equivalence relation. The coarsest such relation is that of numerical equivalence, when two equidimensional cycles are equivalent if their intersection indices with each cycle of complementary dimension coincide. The finest one is the rational (Chow) equivalence, when equivalent cycles are fibres of a family parametrized by a chain of rational curves. Direct product of varieties induces the tensor product structure on the category.

The second step in the definition of the relevant category of pure motives consists in a formal construction of new objects (and relevant morphisms) that are "pieces" of varieties: kernels and images of projectors, *i.e.* correspondences $p : X \to X$ with $p^2 = p$. This produces a *pseudo-abelian*, or *Karoubian* completion of the category. In this new category, the projective line $\mathbb{P}^1$ becomes the direct sum of the (motive of) a point and the Lefschetz motive $\mathbb{L}$ (intuitively corresponding to the affine line).

The third, and last step of the construction, is one more formal enhancement of the class of objects: they now include *all* integer tensor powers $\mathbb{L}^{\otimes n}$, not just non-negative ones, and tensor products of these with other motives. An important role is played by $\mathbb{L}^{-1}$ which is called the Tate motive $\mathbb{T}$.

The first twenty five years of the development of the theory of motives were summarised in the informative Proceedings of the 1991 Research Conference conference "Motives", published in two volumes by AMS in 1994.

Already by that time it was however clear that the richness of ideas and problems involved in this project resists any simple-minded notion of "unification", and with time, the theory of motives was more and more resembling a Borgesian garden of forking paths. Each strand of the initial project tended to unfold in its own direction, whereas the central stumbling stone on the Grothendieck visionary road, the Standard Conjectures, resisted and still resists all efforts.

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The book by Gonçalo Tabuada is a dense combination of a survey paper and a research monograph dedicated to the development of the theory of motives during the next twenty five years. The author contributed many important results and techniques in the theory in recent years. In this book, he focuses on the so called "noncommutative motives". I will make a few brief comments about the scope of this subject.

In very general terms, one can say that motivic constructions of the New Age start not with just smooth varieties but rather with triangulated categories and their enhancements, dg categories. Classical varieties fit there by supplying their derived and more general enhanced derived categories, such as categories of perfect complexes. Enhancement essentially means that morphisms rather than objects are treated as complexes, complexes modulo homotopy, etc. Hence the usual categorical framework is not sufficient anymore: we must deal with 2-categories and eventually with categories of higher level.

Correspondences between such "varieties" are introduced using Morita-like constructions. Recall that in the basic Morita theory morphisms between non-necessarily commutative rings $A \to B$ are replaced with (A, B)-bimodules, and that the difference between commutative and noncommutative rings in this framework essentially vanish because any commutative ring is Morita equivalent to the ring of matrices of any given order over it.

One of the first great surprises of this insight transplanted into (projective) algebraic geometry was Alexander Beilinson's discovery (1983) that the derived category of coherent sheaves of a projective space can be described as a triangulated category made out of modules over a Grassmann algebra. In particular, a projective space became "affine" in some kind of noncommutative geometry! The development of Beilinson's technique led to a general machinery describing triangulated categories in terms of exceptional systems and expanding the realm of candidates to the role of noncommutative motives.

Thus the abstract properties of the categories constructed in this way justify the intuition and terminology of "noncommutative geometry" which was one motivation for M. Kontsevich's project of Noncommutative Motives and became the central subject of Tabuada's book.

This shift of the viewpoint required much work needed to establish how much do we lose by passing from the classical picture to the new one, and what do we gain in understanding of both old and new universes of Algebraic Geometry. Some of these exciting results are surveyed in Tabuada's monograph, and the reader who wants to focus on a particular strand of research will be able to follow the relevant original papers cited in the ample references list.

This stimulating book will be a precious source of information for all researchers interested in algebraic geometry.

Yuri I. Manin

Introduction

The theory of *motives* began in the early sixties when Grothendieck envisioned the existence of a "universal cohomology theory of algebraic varieties" acting as a gateway between algebraic geometry and the assortment of the classical Weil cohomology theories (de Rham, Betti, étale, crystalline). After the release of Manin's foundational article [**Man68**] on the subject, Grothendieck's ideas became popular and a powerful driving force in mathematics.

The theory of *noncommutative motives* is more recent. It began in the eighties when the Moscow school (Beilinson, Bondal, Kapranov, Manin, and others) started the study of algebraic varieties via their derived dg categories of coherent sheaves. It turns out that several invariants of algebraic varieties can be recovered from their derived dg categories. The idea of replacing algebraic varieties by arbitrary dg categories, which are morally speaking "noncommutative algebraic varieties", later led Kontsevich [**Konb**] to envision the existence of a "universal invariant of noncommutative algebraic varieties".

The purpose of this book is to give a rigorous overview of some of the main advances in the theory of noncommutative motives. It is based on a graduate course (18.917 - Noncommutative Motives) taught at MIT in the spring of 2014 and its intended audience consists of graduate students and mathematicians interested in noncommutative motives and their applications. We assume some familiarity with algebraic geometry and with homological/homotopical algebra. The contents of the book can be divided into three main parts:

- Part I: Differential graded categories – Chapter 1.
- Part II: Noncommutative pure motives – Chapters 2-7.
- Part III: Noncommutative mixed motives – Chapters 8-10.

A differential graded (=dg) category is a category enriched over complexes (morphism sets are complexes). An essential example to keep in mind is the derived dg category of an algebraic variety. Several invariants such as algebraic K-theory, cyclic homology (and all its variants), and topological Hochschild homology, can be defined directly on dg categories. In order to study all these invariants simultaneously, we introduce the notion of an *additive invariant* and of a *localizing invariant*.

A functor, defined on the category of (small) dg categories and with values in an additive category, is called an additive invariant if it inverts Morita equivalences and sends semi-orthogonal decompositions in the sense of Bondal-Orlov to direct sums. Chapter 2 is devoted to the study of this class of invariants and to the construction of the *universal* additive invariant. The theory of noncommutative pure motives can be roughly summarized as the study of the target additive category of this universal additive invariant. Making use of Kontsevich's smooth proper dg categories, which are morally speaking the "noncommutative smooth proper algebraic varieties", we introduce in Chapter 4 several additive categories of noncommutative pure motives and relate them to their commutative counterparts. An important example is the category of noncommutative numerical motives. Among other properties, we prove that this category is abelian semi-simple. This result, combined with some noncommutative (standard) conjectures stated in Chapter 5, leads to a (conditional) theory of noncommutative motivic Galois groups and to the extension of the classical theory of intermediate Jacobians to "noncommutative algebraic varieties"; consult Chapters 6 and 7, respectively.

A functor, defined on the category of (small) dg categories and with values in a triangulated category, is called a localizing invariant if it inverts Morita equivalences, preserves filtered homotopy colimits, and sends Drinfeld's short exact sequences to distinguished triangles. The rigorous formalization of this notion requires the language of Grothendieck derivators, which can be found in Appendix A. Chapter 8 is devoted to the study of this class of invariants and to the construction of the *universal* localizing invariant. The theory of noncommutative mixed motives can be roughly summarized as the study of the target triangulated category of the universal localizing invariant. Making use once again of Kontsevich's smooth proper dg categories, we introduce in Chapter 9 several triangulated categories of noncommutative mixed motives and relate them with Voevodsky's triangulated category of geometric mixed motives. Finally, in Chapter 10 we briefly describe the (unconditional) theory of noncommutative motivic Hopf dg algebras, which is the mixed analogue of the theory of noncommutative motivic Galois groups.

Although recent, the theory of noncommutative motives already led to the (partial) solution of some open problems and conjectures in adjacent research areas. Some of these will be discussed throughout the book.

We refrain from giving a lengthy introduction to the contents of each chapter. The table of contents combined with the introduction of each chapter provides the corresponding information.

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