1 Brief note for Borel-Moore homology

1.1 Definition of Borel-Moore homology

In this section, we define so-called Borel-Moore homology and look for its properties. It is similar to the original homology, and in fact coincides with it if the given space is compact. It provides a very important tool as we proceed our argument.

Definition-Theorem 1.1. Let $X$ be a finite CW-complex. We assume that we can embed $X$ into a $C^\infty$-mainfold $M$ which is countable at infinity, and furthermore we assume this embedding is a (proper) neighborhood retract. Then the following objects are equivalent. (Coefficients are assumed to be in any field of char 0.)

(a) $H_*(\hat{X}, \infty)$, where $\hat{X} = X \cup \{\infty\}$ is the one-point compactification of $X$.

(b) $H_*(\tilde{X}, \tilde{X} \setminus X)$, where $\tilde{X}$ is an arbitrary compactification of $X$ and $(\tilde{X}, \tilde{X} \setminus X)$ is a CW pair.

(c) $H_*(C^\infty_{BM}(X), d)$, where $C^\infty_{BM}(X)$ is a chain complex of locally finite sums of singular chains (but sum itself can be infinite.)

(d) $H^{m-*}(M, M \setminus X)$, where $M$ is a smooth oriented manifold of dimension $m$ and there exists an embedding $X \hookrightarrow M$ as before.

(d') $H^{m-*}(X)$ if $X$ itself is a smooth oriented manifold of dimension $m$.

(e) $H_*(D_*(X), d)$. Assume there exists an embedding $X \hookrightarrow M$ to a smooth manifold $M$ of dimension $m$. Then we define $D_*(X)$ be a chain complex of distributions supported on $X$. (A distributions of degree $k$ is a continuous linear function $\phi : \Omega^{m-k}_c \to \mathbb{R}$ defined on the space of $(m-k)$-forms with compact support.)

Now we define $H^\infty_{BM}(X)$, the Borel-Moore homology on $X$, as any of these notions.

Remark. From now on, we follow the notation of [CG], and write $H_*$ for $H^{BR}_*$ and $H^\text{ord}_*$ for $H_*$.
1.2 Properties of Borel-Moore homology

Here are some properties of Borel-Moore homology, which we admit without proof.

(a) Proper push-forward. If $f : X \to Y$ is proper, then $f_* : H_*(X) \to H_*(Y)$ is defined by

$$f_* : H^{ord}_*(\bar{X}) \to H^{ord}_*(\bar{Y}).$$

(Note that $f$ is continuous at $\infty$.)

(b) Long exact sequence. Let $F \subset X$ be a closed subset of $X$ and $U = X \setminus F$. Then we have a long exact sequence of Borel-Moore homology groups

$$\cdots \to H_n(F) \to H_n(X) \to H_n(U) \to H_{n+1}(F) \to \cdots$$

defined as follows. Let $X \hookrightarrow M$ be a closed embedding to a smooth (oriented) manifold of dimension $m$, then we have

$$\cdots \to H^{m-n}(M, M \setminus F) \to H^{m-n}(M, M \setminus X) \to H^{m-n}(M, M \setminus U) \to H^{m-n}(M, M \setminus F) \to \cdots.$$

To convert this to the desired sequence, the only interesting part is $H^{m-n}(M, M \setminus U)$. However, we may find an open subset $M' \subset M$ such that $X \cap M' = U$. Then $H^{m-n}(M, M \setminus U) = H^{m-n}(M', M' \setminus U) = H_n(U)$ by excision theorem.

(c) Fundamental class. If $X$ is a smooth oriented manifold of dimension $m$, we define the fundamental class of $X$, denoted by $[X] \in H_m(X)$, which corresponds to $[\bar{X}] \in H^ord_m(\bar{X})$. In particular, if $X$ is an irreducible complex algebraic variety, then we define $[X]$ by the image of $[X^{reg}] \in H_n(X^{reg})$ under the isomorphism $H_n(X) \xrightarrow{\sim} H_n(X^{reg})$ induced by the long exact sequence since $\dim(X \setminus X^{reg}) \leq m - 1$. Furthermore, if $X$ is not irreducible, then define $[X] = \sum [X_i]$ where $X_i$’s are irreducible components of $X$.

(d) Intersection pairing. For two closed subsets $Z, Z' \subset M$ of a smooth oriented manifold $M$ of dimension $m$, which are neighborhood retracts, define

$$\cap : H_i(Z) \times H_j(Z') \to H_{i+j-m}(Z \cap Z')$$

as the map induced by

$$\cup : H^{m-i}(M, M \setminus Z) \times H^{m-j}(M, M \setminus Z') \to H^{2m-i-j}(M, M \setminus (Z \cap Z')).$$

(e) Poincaré duality. Use the same convention above. From the cup product, i.e.,

$$\cup : H^c_{m-i}(M, M \setminus Z) \times H^c_{m-j}(M, M \setminus Z') \to H^{2m-i-j}_c(M, M \setminus (Z \cap Z')),$$

($H^c_*$ is the cohomology group with compact support) we derive the following pairing

$$\cap : H^{ord}_i(Z) \times H^{ord}_j(Z') \to H^{ord}_{i+j-m}(Z \cap Z').$$
via the isomorphism \( H_{m-i}^\overline{\text{ord}}(M, M \setminus Z) \simeq H_i^\text{ord}(Z) \). In particular, by substituting \( i \) with \( m-j \) and \( Z, Z' \) with \( M \) we obtain
\[
\cap : H_{m-j}^\overline{\text{ord}}(M) \times H_j(M) \to H_0^\overline{\text{ord}}(M) \simeq \mathbb{C},
\]
which is nondegenerate. Thus it gives a natural isomorphism \( H_j(M) \simeq H_{m-j}^\overline{\text{ord}}(M)^\vee \simeq H^{m-j}(M) \) where the latter map is derived from Poincaré duality.

(f) Kunneth formula. Let \( M_1 \) and \( M_2 \) be CW complexes. By the usual Kunneth formula, i.e.,
\[
H^s_\overline{\text{ord}}(M_1, M_1 \setminus M_1) \otimes H^s_\overline{\text{ord}}(M_2, M_2 \setminus M_2) \simeq H^s_\overline{\text{ord}}(M_1 \times M_2, M_1 \times M_2 \setminus M_1 \times M_2)
\]
we have an isomorphism
\[
\boxtimes : H_\ast(M_1) \otimes H_\ast(M_2) \simeq H_\ast(M_1 \times M_2)
\]

(g) Restriction with supports. Let \( i : N \hookrightarrow M \) be a closed embedding of oriented manifolds. Then for \( Z \subset M \), we define
\[
i^\ast : H_\ast(Z) \to H_{\ast - \text{codim}_MN}(Z \cap N) : c \mapsto c \cap [N].
\]
Note that it depends on the embedding \( i \).

(h) Diagonal Reduction. Let \( M \) be an oriented manifold, and \( Z, Z' \subset M \). Also, let \( \Delta : M \hookrightarrow M \times M \) be a diagonal map. Then we have a natural identity
\[
c \cap c' = \Delta^\ast(c \boxtimes c') = (c \boxtimes c') \cap [\Delta(M)], \quad \text{for } c \in H_\ast(Z) \text{ and } c' \in H_\ast(Z').
\]

(i) Smooth pullback. Let \( X \) be locally compact and \( p : \tilde{X} \to X \) be a fiber bundle with smooth oriented fiber \( F \) of dimension \( d \). Also assume \( p \) is oriented, that is transition functions are orientation preserving. Then we have a natural pullback
\[
p^\ast : H_\ast(X) \to H_{\ast + d}(\tilde{X}),
\]
which is locally \( c \mapsto c \boxtimes [F] \). Also, if we are in the following situation,
\[
\begin{array}{ccc}
  F & \longrightarrow & \tilde{X} \\
  & \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
(j) Projection formula. Let $p : \tilde{M} \to M$ be a oriented fiber bundle over a smooth oriented variety $M$ and $Z \subset M$, $Z' \subset \tilde{M}$ are closed. Also, assume $p^{-1}(Z) \cap Z' \to M$ is proper. Then we have

$$p_*(p^*c \cap c') = c \cap (p_*c') \in H_*(p(p^{-1}(Z) \cap Z'))$$

for $c \in H_*(Z)$ and $c' \in H_*(Z')$.

(k) Specialization. Let $S$ be a smooth manifold and $o \in S$. Now choose a neighborhood of $o \in U \subset S$ and identify $(o, U) = (0, \mathbb{R}^n)$. Also assume $\pi : Z \to S$ is given and $\pi : \pi^{-1}(S \setminus \{o\}) \to S \setminus \{o\}$ is a fiber bundle with fiber $F$. Then by Kunneth formula, and by shrinking $U$ if necessary, there exists a chain of isomorphisms

$$H_*(\pi^{-1}(\mathbb{R}_{>0} \times \mathbb{R}^{n-1})) \xrightarrow{\cong} H_{*-n}(F) \otimes H_n(\mathbb{R}_{>0} \times \mathbb{R}^{n-1})$$

$$\xrightarrow{\cong} H_{*-n}(F) \otimes H_1(\mathbb{R}_{>0}) \xrightarrow{\cong} H_{*-n+1}(\pi^{-1}(\mathbb{R}_{>0}))$$

with an appropriate degree shift, where $\mathbb{R}_{>0}$ denotes the positive part of the line which corresponds to the first coordinate. Then we have the following map

$$H_*(\pi^{-1}(S \setminus \{o\})) \xrightarrow{res} H_*(\pi^{-1}(\mathbb{R}_{>0} \times \mathbb{R}^{n-1})) \xrightarrow{\cong} H_{*-n+1}(\pi^{-1}(\mathbb{R}_{>0})) \xrightarrow{\partial} H_{*-n}(\pi^{-1}(o))$$

where the last map is derived from the long exact sequence

$$\cdots \to H_*(\pi^{-1}(o)) \to H_*(\pi^{-1}(\mathbb{R}_{\geq 0})) \to H_*(\pi^{-1}(\mathbb{R}_{>0})) \xrightarrow{\partial} H_{*-1}(\pi^{-1}(o)) \to \cdots.$$ 

(l) Cohomology action.

(m) Thom isomorphism.

(n) Access Intersection formula.

### 1.3 Convolution in Borel-Moore homology

All we stated in previous sections ready us to define a convolution algebra of Borel-Moore homology on a space. Let $M_1, M_2, M_3$ be connected oriented $C^\infty$-manifold, and $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$ be closed subsets. Suppose we have the following diagram:

$$
\begin{array}{ccc}
Z_{12} \subset M_1 \times M_2 & M_1 \times M_2 & M_1 \times M_3 \\
p_{12} & \leftarrow & \rightarrow & \rightarrow & \rightarrow \\
 & M_2 \times M_3 \supset Z_{23} & & & \\
p_{13} & p_{23} & \\
\end{array}
$$

Also, we assume that $p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$ is proper. Then define $Z_{12} \circ Z_{23} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$. By assumption, it is closed in $M_1 \times M_3$. Now we define a convolution ($d = \dim M_2$)

$$*: H_1(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23})$$

$$(c_{12}, c_{23}) \mapsto (p_{13})_*(p_{12}^*c_{12} \cap p_{23}^*c_{23}) = (p_{13})_*((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})).$$
This is in fact associative. Now we give some special examples.

(a) $M_1 = M_2 = M_3 = M$. Assume $Z_{12}, Z_{23} \subset M$ are two closed subsets and consider these as the subset of $M_{12}, M_{23}$, respectively, via the diagonal morphism. Then $Z_{12} \circ Z_{23} = Z_{12} \cap Z_{23}$, and the convolution coincides with the intersection pairing.

(b) $M_1 = \{\ast\}$. Let $f : M_2 \rightarrow M_3$ be a proper map of connected varieties, $Z_{12} = \{\ast\} \times M_2$, and $Z_{23} = \text{Graph}(f)$. Then $Z_{12} \circ Z_{23} = \text{im} f$, and for $c \in H_\ast Z_{12}, c \ast [\text{Graph}(f)] = f_\ast (c)$. In other words, the convolution is interpreted as a proper push-forward.

(c) $M_3 = \{\ast\}$. Let $f : M_1 \rightarrow M_2$ be a smooth map of oriented connected manifolds, $Z_{12} = \text{Graph}(f)$, and $Z_{23} = M_2 \times \{\ast\}$. Then $Z_{12} \circ Z_{23} = M_1$, and for $c \in H_\ast Z_{23}, [\text{Graph}(f)] \ast c = f^\ast (c)$. In other words, the convolution is interpreted as a smooth pullback.

Now we define a convolution algebra structure as follows.

**Definition 1.2.** Assume $\pi : M \rightarrow N$ be a proper map, where $M$ is a smooth complex manifold of (real) dimension $m$ and $N$ be a variety. Then by putting $M_1 = M_2 = M_3 = M$ and $Z = Z_{12} = Z_{23} = M \times_N M$, we have the convolution

$$H_\ast(Z) \times H_\ast(Z) \rightarrow H_\ast(Z \circ Z) = H_\ast(Z).$$

We call $(H_\ast(Z), +, \ast)$ a convolution algebra.

**Proposition 1.3.** $(H_\ast(Z), +, \ast)$ is an associative algebra. It has a unit $[\Delta(M)] \in H_m(Z)$. Furthermore, $H_\ast(Z)$ is a subalgebra of $H_\ast(Z)$ and it acts on $H_\ast(Z)$ by degree preserving multiplication.

Also, this algebra has natural representations as follows.

**Proposition 1.4.** Let $M_x = \pi^{-1}(x)$ for $x \in N$. Then by setting $M_1 = M_2 = M$, $M_3 = \{\ast\}, Z = Z_{12} = M \times_N M$, and $Z_{23} = M_x$, we see that $Z \circ M_x = M_x$. Then the convolution map now gives the natural left $H_\ast(Z)$-module structure on $H_\ast(M_x)$. Also, $H_m(Z)$ acts on $H_\ast(M_x)$ with preserving degrees. Similarly the natural right action on $H_\ast(M_x)$ is also well-defined.

We state the following proposition for future use.

**Proposition 1.5.** Let $X$ be a complex variety of complex dimension $n$, and $X_1, \cdots, X_m$ be the irreducible components of $X$ of the highest dimension. Then $H_{2n}(X)$ has a basis $[X_1], \cdots, [X_m]$. 

5
2 Preliminaries on Lie Theory and Linear Algebraic Groups

2.1 Lie Theory

In this section, we deal with some notions and properties of Lie algebras and Lie groups. We assume that audiences (and readers) are familiar with "basic" properties of Lie algebras, such as solvable and nilpotent Lie algebras, semisimple and nilpotent elements, root space decomposition, finite dimensional representations of \( \mathfrak{sl}_2 \), etc. If not, there will be a long journey waiting for you...

From now on, to simplify argument we assume that all the Lie groups and Lie algebras in this section is semisimple and over \( \mathbb{C} \). Then the next theorem says we are on a good track.

**Theorem 2.1.** A complex semisimple Lie group is holomorphically isomorphic to a complex Lie group of matrices.


Now we define a notion of Borel subgroup.

**Definition 2.2.** A Borel subgroup \( B \) of Lie group \( G \) is a maximal solvable connected Lie subgroup.

**Proposition 2.3.** Here are some properties of Borel subgroups.

(a) There exists a Levi decomposition of \( B \), namely, \( B = TU \) where \( T \) is a maximal torus of \( G \) contained in \( B \) and \( U \) is a unipotent radical of \( B \).

(b) \( N_G(B) = B \).

(c) All Borel subgroups are conjugates to one another.

**Proof.** Refer to [HumLAG], Chapter 21 to 23.

This theorem will be used on proving Bruhat decomposition.

**Lemma 2.4.** \( N(T) \cap B = T \).

**Proof.** By [HumLAG] Proposition 19.4, \( N_B(T) = C_B(T) = T \).

Now, we define an abstract Weyl group. First of all, there exists an abstract Cartan subalgebra \( \mathcal{H} \), which is identified with \( \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \) for any Borel subalgebra \( \mathfrak{b} \). Indeed, there exists a canonical isomorphism \( \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{b'}/[\mathfrak{b'}, \mathfrak{b'}] \) by adjoint operation. Now, for a pair \( \mathfrak{h} \subset \mathfrak{b} \) of a Cartan subalgebra and a Borel subalgebra, we have a canonical identification \( \mathfrak{h} \leftrightarrow \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathcal{H} \). Then the action of Weyl group \( W_T \) on \( \mathfrak{h} \) is transferred to an action on \( \mathcal{H} \). Now,
**Definition 2.5.** We define the abstract Weyl group $W$ as $W_T$ for any $T$ with the action on $\mathcal{H}$.

Then we can define the relative position between two Borel subalgebras, namely,

**Definition 2.6.** $R(b, b') \in W$, by an element $w \in W$ such that $wbw^{-1} = b'$.

One can show that it is well defined, that is independent of choice of a Cartan subalgebra $h \subset b \cap b'$.

Now we define a regular element in a Lie algebra.

**Definition 2.7.** An element $x \in \mathfrak{g}$ is regular if $\dim \mathcal{Z}(x)$ is minimal. We call such a minimal dimension the rank of $\mathfrak{g}$, denoted by $\text{rk } \mathfrak{g}$. Also, $\mathfrak{g}^{sr}$ is the set of all semisimple regular elements in $\mathfrak{g}$.

**Proposition 2.8.** Here are some properties of regular elements.

(a) For any $x \in \mathfrak{g}$, the characteristic polynomial of the adjoint action of $x$, namely $P_x(T) = \det(T \cdot \text{id} - \text{ad}x)$, has a zero at $T = 0$ with a multiplicity $\geq \text{rk } \mathfrak{g}$.

(b) $x \in \mathfrak{g}^{sr}$ if and only if the coefficient of $T^{\text{rk } \mathfrak{g}}$ in $P_x(T)$ is nonzero.

(c) $\mathfrak{g}^{sr}$ is a $G$-stable Zariski distinguished open subset of $\mathfrak{g}$.

(d) For any $x \in \mathfrak{g}^{sr}$, $\mathcal{Z}(x)$ is a Cartan subalgebra of $\mathfrak{g}$. Conversely, any Cartan subalgebra contains such an element.

**Proof.** The first statement is obvious since $\dim \ker \text{ad}x \geq \text{rk } \mathfrak{g}$. Proof of the second statement exploits the Jordan decomposition of $x$. The third one directly follows from the second statement, since the coefficient of $T^{\text{rk } \mathfrak{g}}$ in $P_x(T)$ is a polynomial of coordinates of $x$, which is $G$-stable since adjoint actions stabilize a characteristic polynomial. The last statement uses the fact that a Cartan subalgebra is a minimal Engel subalgebra, which is the generalized eigenspace of $\text{ad}x$ corresponding to eigenvalue 0 for some $x \in \mathfrak{g}$. \qed

There also exists the notion of a regular element in an (abstract) Cartan subalgebra. That is, if the cardinality of the orbit of $x \in h$ (resp. $x \in \mathcal{H}$) under the action of (abstract) Weyl group is the same as that of (abstract) Weyl group, we say that $x$ is regular. In other words, $x$ is regular if and only if it does not belong to any hyperplane perpendicular to a root. Also, we define $h^{\text{reg}}$ (resp. $\mathcal{H}^{\text{reg}}$) as the set of regular elements. Indeed, the usage of the same term in both cases is not an accident as the following proposition indicates.

**Proposition 2.9.** $x$ is regular if and only if $x$ is (semisimple) regular.

**Proof.** Exercise. \qed

**Corollary 2.10.** $h^{\text{reg}}$ (resp. $\mathcal{H}^{\text{reg}}$) is $W_T$ (resp. $W$)-stable Zariski open subset of $h$ (resp. $\mathcal{H}$).

**Proof.** One can also prove this by just using the definition, since the union of hyperplanes each of which is perpendicular to some root is stable. \qed
2.2 Trivia on Linear Algebraic Groups

Here are some useful theorems for the future.

**Theorem 2.11.** Let $X$ be a quasi-affine variety equipped with an action of a unipotent affine algebraic group. Then its orbits are closed in $X$.

*Proof.* [HumLAG] Exercise 17.8.

**Theorem 2.12.** (Borel) If a connected solvable group acts on a proper nonempty variety, then it has a fixed point.


**Theorem 2.13.** (Equisingularity of orbits) Let $X$ be a variety with an action of an algebraic group $G$. Then the following holds.

1. Every $G$-orbit is locally closed and smooth.
2. The boundary of an orbit is the union of other orbits with lower dimensions.

*Proof.* Let $Y \subset X$ be an orbit of this action. Then by the theorem of Chevalley, it is constructible, thus $Y$ contains a (relative) open subset $U$ of $\bar{Y}$. Then since $G \cdot U = Y$, we conclude that $Y$ is closed in $\bar{Y}$, or $Y$ is locally closed. Also, since $Y$ is dense in $\bar{Y}$, $Y$ has a smooth point of $\bar{Y}$. (Smooth points comprise an open subset of $\bar{Y}$.) Since $Y$ is an orbit, all points in $Y$ are "equivalent", which means that they are all smooth in $\bar{Y}$. Now, $\bar{Y} \setminus Y$ is closed and $G$-stable by definition, thus consists of $G$-orbits. Note that their dimension is strictly less than that of $Y$, since $Y$ is open dense in $\bar{Y}$.

**Theorem 2.14.** An algebraic group is connected if and only if it is irreducible.

*Proof.* One way is trivial. Now suppose $e \in G$ is in two irreducible components in $G$, say $X$ and $Y$. Then $e \in XY \subset G$ is irreducible and contains $X$ and $Y$. Thus $X = Y = X \times Y$. In general it follows from free $G$-action on $G$.

Now let $X$ be a smooth complex projective variety and $\mathbb{C}^*$ act on $X$. By Theorem 2.12, we know that the set of fixed point $W \subset X$ is nonempty. Actually, we can say more; fixed points on $X$ are determined by the following lemma.

**Lemma 2.15.** For any $x \in X$, $\lim_{t \to 0} t \cdot x$ and $\lim_{t \to \infty} t \cdot x$ exist, and they are fixed points. Thus, indeed $\{ \lim_{t \to 0} t \cdot x | x \in X \}$ is the set of all fixed points on $X$.

*Proof.* Omitted.
Now for each $w \in W$, we define the attracting set
$$X_w = \{x \in X | \lim_{t \to 0} t \cdot x = w\}.$$ 

Also, we define the $C^*$ action on $T_wX$ by induced action, which makes sense because $w \in W$. Now if we assume $W$ to be discrete, then $T_wX = T_w^+X \oplus T_w^-X$, where $T_w^+X = \bigoplus_{n \in \mathbb{Z}^+} T_wX(n)$ and $T_w^-X = \bigoplus_{n \in \mathbb{Z}^-} T_wX(n)$ are weight space decompositions. Note that all the weights are integers since $\text{Hom}_{\text{alg}}(C^*, C^*) = \{z \to z^n | n \in \mathbb{Z}\}$, and $T_wX$ does not have a weight 0 subspace since $W$ is assumed to be discrete. (Otherwise there should exist a curve of fixed points passing through $w$ in that direction.) Now we have the following theorem.

**Theorem 2.16.** *(Bialynicki-Birula Decomposition)* Assume $W$ is finite. Then,

(a) $X = \bigsqcup_{w \in W} X_w$, where $X_w$ are locally closed and smooth.

(b) There exist natural $C^*$-equivariant isomorphisms of varieties $X_w \simeq T_w(X_w) = T_w^+X$.

**Proof.** Omitted.

3 Springer Resolution and Steinberg Variety

In this section, $G$ is always a connected semisimple connected Lie group.

3.1 Bruhat Decomposition

**Theorem 3.1.** *(Bruhat Decomposition)* Fix a Borel pair $(B,T)$ and its Lie algebra $\mathfrak{b}$. There exist following bijections

$$
\begin{align*}
W_T & \to B \backslash G / B \to \{B\text{-orbits on } B\} \to \{G\text{-diagonal orbits on } B \times B\} \\
\omega & \mapsto B\omega B \mapsto [\omega \mathfrak{b} \omega^{-1}] \mapsto [(\mathfrak{b}, \omega \mathfrak{b} \omega^{-1})].
\end{align*}
$$

**Sketch of proof.** The second bijection is obvious by the identification $G/B \simeq B$. To justify the first bijection, we proceed the following steps.

- Show that the fixed points of $T$-action on $B$ is of the form $w \mathfrak{b} w^{-1}$ for $w \in W_T$. Note that they are all different from one another since $N_G(B) = B$ and by Lemma 2.4.

- Apply Bialynicki-Birula decomposition to set $\mathcal{B} = \sqcup_{w \in W} \mathcal{B}_{w \mathfrak{b} w^{-1}} \simeq \sqcup_{w \in W} \mathcal{B}_w$.

- Prove that $U \cdot w \subset \mathcal{B}_w$ where $U = R_\alpha(B)$, and $\dim \mathcal{B}_w = \dim U \cdot w$.

- By using Theorem 2.11, conclude that $U \cdot w = U \cdot T \cdot w = B \cdot w = \mathcal{B}_w$.

The last bijection is rather straightforward. \qed
Remark. One can identify $\mathbb{W} \simeq W_T$ and get a bijection without choosing $B$ and $T$.

The main goal of this section is to understand the following diagram. We will not deal with all the details of the proof, and focus on resolving this seemingly complicated diagram into our down-to-earth language.

**Theorem 3.2.** (Grothendieck’s simultaneous resolution, and Springer resolution) There exists the following commutative diagram

\[
\begin{array}{ccccccc}
\tilde{N}^{\text{reg}} & \rightarrow & \tilde{N}^G & \rightarrow & \tilde{g} & \rightarrow & \mu^{-1}(g^{sr}) = \nu^{-1}(H^{reg}) \\
\downarrow & & \downarrow & & \downarrow & & \\
N^{\text{reg}} & \rightarrow & N^G & \rightarrow & g & \rightarrow & H^{reg} \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{H}/\mathbb{W} & \rightarrow & \mathcal{H}^{\text{reg}}/\mathbb{W} & \rightarrow & \\
\end{array}
\]

where
- $\tilde{g} = \{(x, b) \in g \times B | x \in b\}$.
- $\mathcal{H}$ is the abstract Cartan algebra.
- $\mathbb{W}$ is the abstract Weyl group.
- $N$ is the nilpotent cone.
- $\rho : g \rightarrow \mathcal{H}/\mathbb{W}$ is induced by $C[\mathcal{H}/\mathbb{W}] \simeq C[\mathcal{H}]^{\mathbb{W}} \simeq C[g]^G \hookrightarrow C[g]$.
- $\mu(x, b) = x, \nu(x, b) = x \mod [b, b]$.

Furthermore, $\tilde{g}^{sr} \rightarrow \tilde{g}^{sr}$ is a principal $\mathbb{W}$-bundle.

Now we justify this diagram. First of all, in order to construct the morphism $\rho$ we need some theorems (and a lemma).

**Theorem 3.3.** (Chevalley) $C[g]^G \simeq C[h]^W$.

**Theorem 3.4.** (Chevalley-Shephard-Todd) Assume a finite group $G$ acts linearly on $V$. Then $C[V]^G$ is a free polynomial algebra if and only if $G$ is generated by ”pseudoreflections.”

Algebraic version of Riemann’s removable singularity theorem needed? not sure
Lemma 3.5. $\mathcal{W}$-invariant polynomials separates $\mathcal{W}$-orbits on $\mathcal{H}$.

Now, first of all, we have $\mathcal{H} \to \text{Spec} \mathbb{C}[\mathcal{H}]^W$ and $\mathcal{H}/\mathcal{W} \to \text{Spec} \mathbb{C}[\mathcal{H}]^W$. Then the lemma above show that it is injective. Also, it is surjective since $\mathbb{C}[\mathcal{H}]^W \hookrightarrow \mathbb{C}[\mathcal{H}]$ is finite; see [Harris, p. 124-125] for details. Then by Theorem 3.3 $\rho$ is well defined. Note that Theorem 3.4 shows that $\mathcal{H}/\mathcal{W}$ is also a vector space.

Then we can easily see that the main square in the middle is well justified. In fact, $\rho$ is constructed so as to make the diagram commute. Also, the square on the right shows the generic nature of this resolution; $g^* \to g$ is the pullback of $\mathcal{H}^{reg} \to \mathcal{H}^{reg}/\mathcal{W}$, which is a principal $\mathcal{W}$-bundle.

Now we focus on the left hand side. In fact, there is another canonical map, namely $\tilde{g} \to B$. By investigating this map, we obtain the following result.

Theorem 3.6. $\pi : \tilde{g} \to B$ is a $G$-equivariant vector bundle with fiber $b$, and $G$-equivariant isomorphic to $G \times_B b \to G \times_B \{e\} \simeq B$ via $G \times_B b \to \tilde{g} : (g,x) \mapsto (gxg^{-1},gbg^{-1})$. Furthermore, if we restrict this argument to $\tilde{N}$, we obtain a $G$-equivariant isomorphism $G \times_B n \to \tilde{N}$ and a $G$-equivariant vector bundle with fiber $n$.

Theorem 3.7. $\tilde{N} \to N$ is surjective. In other words, for any nilpotent $x \in g$, there exists a Borel subalgebra containing $x$.

Proof. Omitted. □

Remark. If we identify $n \simeq b^\perp$ via the Killing form of $g$, we also obtain an isomorphism $T^*B \simeq G \times_B b^\perp \simeq G \times_B n \simeq \tilde{N}$. One interesting fact (which should be true in the just world of mathematics) is that $\mu : \tilde{N} = T^*B \to N$ is the moment map with respect to the natural $G$-action on $T^*B$.

Meanwhile, $N = \rho^{-1}(0)$ is equivalent to the following

Lemma 3.8. $x \in g$ is nilpotent if and only if $P(x) = 0$ for all $P \in \mathbb{C}[g]^G = \mathbb{C}[h]^W$ satisfying $P(0) = 0$.

Proof. Omitted. □

Now we figure out some properties of $\tilde{N} \to N$. First of all, the remark above says that $\tilde{N} = T^*B$ is smooth and connected, hence irreducible. Then by Theorem 3.7 $N$ is irreducible.

Proposition 3.9. $N^{reg}$ is a Zariski open dense single $G$-orbit of $N$.

To prove this, we use the fact that $N$ consists of finite $G$-orbits. We prove this later. Then on this open subset, the situation is quite simple, since

Proposition 3.10. Any regular nilpotent element is contained in a unique Borel subalgebra.
Thus $\tilde{N}^{reg} \to N^{reg}$ is an isomorphism, and we obtain the following cliffhanger;

**Theorem 3.11.** (Springer) $\tilde{N} \to N$ is a resolution of singularity.

For future use, we state (and not prove) the following theorem.

**Theorem 3.12.** The (finite) orbits on $N$ comprise an algebraic stratification of $N$.

### 3.2 Steinberg Variety

**Definition 3.13.** The Steinberg variety is $Z = \tilde{N} \times_N \tilde{N}$.

Then we have a morphism $Z \to \tilde{N} \times \tilde{N} \simeq T^*B \times T^*B \simeq T^*(B \times B)$. Note that the second identity contains the "twisting" of sign; that is we send $(\omega_1, b_1, \omega_2, b_2)$ to $(\omega_1, b_1, -\omega_2, b_2)$. By this identification we see that the image of $T^*B \xrightarrow{\Delta_{reg}} T^*B \times T^*B \simeq T^*(B \times B)$ becomes the conormal bundle of $\Delta B$ with respect to the inclusion $B \xrightarrow{\Delta_B} B \times B$. In fact, with this identification we can say more.

**Proposition 3.14.** $Z$ is the union of the conormal bundles to all $G$-orbits in $B \times B$.

**Proof.** Following the definition of morphisms above to get

\[
Z = \{(x, b_1, x, b_2) | x \in N \cap b_1 \cap b_2 \}
= \{(g_1, x, g_2, x) | x \in n, g_1^{-1}g_2 \in G_x \} \subset (G \times_B n) \times (G \times_B n)
= \{(g_1, x^*, g_2, x^*) | x^* \in b_1^\perp, g_1^{-1}g_2 \in G_{x^*} \} \subset (G \times_B b_1^\perp) \times (G \times_B b_1^\perp)
= \{(g_1x^*g_1^{-1}, g_1bg_1^{-1}, g_2x^*g_2^{-1}, g_2bg_2^{-1}) | x^* \in T_{b_1}^*B \simeq b_1^\perp, g_1^{-1}g_2 \in G_{x^*} \} \subset (T^*B) \times (T^*B)
= \{(x^*, g_1bg_1^{-1}, x^*, g_2bg_2^{-1}) | x^* \in g_1b_1^\perp g_1^{-1} \cap g_2b_2^\perp g_2^{-1} \subset g^* \} \subset (T^*B) \times (T^*B)
= \{(x^*, b_1, -x^*, b_2) | x^* \in b_1^\perp \cap b_2^\perp \subset g^* \} \subset T^*(B \times B)
= \{(x_1^*, b_1, x_2^*, b_2) | x_1^* \in b_1^\perp, x_2^* \in b_2^\perp, x_1^*(y) + x_2^*(y) = 0 \ \forall y \in g \}
= \{(x_1^*, b_1, x_2^*, b_2) | (x_1^*, x_2^*) \in T_{(b_1, b_2)}^*(B \times B) \simeq (g \times b_1 \times b_2)^*, (x_1^*, x_2^*)(y + b_1, y + b_2) = 0 \ \forall y \in g \}
\]

The description of the last line means that $(x_1^*, x_2^*)$ kills all tangent vectors corresponding to the infinitesimal $G$-action on $(b_1, b_2)$, which are exactly the tangent vectors on the $G$-orbit containing that point. \[ \square \]

Now, write $B \times B = \sqcup_{w \in W} Y_w$, where $Y_w = [(b, wbw^{-1})]$ is the orbit which corresponds to $w \in W$ by Bruhat decomposition. Then the proposition above tells that $Z = \sqcup_{w \in W} C_{B \times B/Y_w}$. Then since each $C_{B \times B/Y_w}$ is irreducible and has the same dimension, (and also locally closed,) every irreducible component of $Z$ is the closure of $Y_w$ for a unique $w \in W$. do we need locally closedness?
Meanwhile, there is another natural map \( Z \to \mathcal{N} \), and we can partition \( Z \) into \( \sqcup Z_\mathcal{O} \), where \( \mathcal{O} \subset \mathcal{N} \) is a nilpotent \( G \)-orbit and \( Z_\mathcal{O} \) is its preimage. It turns out to be a good partition, as the next theorem indicates (which we will not prove.)

**Theorem 3.15.** Every irreducible component of \( Z_\mathcal{O} \) is equidimensional to \( Z \).

It means that the closure of an irreducible component of \( Z_\mathcal{O} \) for some \( \mathcal{O} \) is an irreducible component of \( Z \). Now for \( x \in \mathcal{N} \), we define \( B_x = \{(x, b) \mid x \in b\} \), and let \( B_x = \cup_i B^i_x \) where \( B^i_x \) is an irreducible component of \( B_x \). Then we see that \( Z_\mathcal{O} = \tilde{\mathcal{O}} \times \mathcal{O} = G \times_{G_x} (B_x \times B_x) = \cup_{i,j} G \times_{G_x} (B^i_x \times B^j_x) \) via the identification \( \tilde{\mathcal{O}} = G \times_{G_x} B_x \) for some \( x \in \mathcal{O} \), we conclude that any irreducible component of \( Z \) is the closure of \( G \times_{G_x} (B^i_x \times B^j_x) \) for some \( x, i, j \). Also from this argument the next theorem follows.

**Theorem 3.16.** Every irreducible component of \( B_x \) is equidimensional to \( B_x \), and the dimension is equal to \( \dim B - \frac{1}{2} \dim \mathcal{O} \).

Furthermore, \( G \times_{G_x} (B^i_x \times B^j_x) = G \times_{G_x} (B''^i_x \times B''^j_x) \) if and only if \( (B^i_x, B^j_x) \) is conjugate to \( (B''^i_x, B''^j_x) \) by \( G_x \), or equivalently, by the component group \( C_x \) of \( G_x \). Now, we prove the following theorem, to which we procrastinate.

**Theorem 3.17.** Nilpotent \( G \)-orbits are finite.

**Proof.** \( Z \) has finitely many irreducible components, and \( Z_\mathcal{O} \) has at least one irreducible component whose closure is an irreducible component of \( Z \) which is different from the closure of any other such one. \( \square \)

## 4 Springer Correspondence

### 4.1 Identification of Convolution Algebra

Consider the construction of convolution algebra with respect to Borel-Moore homology. Using Proposition 1.3, we see that \( H_*(Z) \) is naturally equipped with a convolution algebra structure. Furthermore, the top dimensional part \( H_m(Z) \) where \( m = \dim_{\mathbb{R}} \tilde{\mathcal{N}} \) is a subalgebra of \( H_*(Z) \), with the unit \( \Delta \tilde{\mathcal{N}} \subset Z \). It has a basis \( \{[\overline{C_{B \times Y}}]\}_{w \in \mathcal{W}} \). Now we state a fundamental theorem, whose proof we omit.

**Theorem 4.1.** There is a canonical algebra isomorphism \( \phi : \mathbb{Q}[\mathcal{W}] \xrightarrow{\cong} H_m(Z) \).

We would rather explain this isomorphism than prove this. Unfortunately, this isomorphism does not identify \( [\overline{C_{B \times Y}}] \) with \( w \). Let \( T_w \) denote \( [\overline{C_{B \times Y}}] \). Also, we define a partial order, also known as Bruhat order, on \( \mathcal{W} \) as follows.

**Definition 4.2.** (Bruhat order) For \( y, w \in \mathcal{W} \), we write \( y \leq w \) if \( Y_y \subset Y_w \).
Remark. For the reader who is familiar with a Coxeter group, note that it is equivalent to the Bruhat order on a Coxeter group. In other words, \( y \leq w \) if and only if some (or any) reduced word of \( w \) contains a substring which is a reduced word of \( y \). Indeed, the choice of a Borel subalgebra is equivalent to the choice of simple roots, and it makes \( \mathcal{W} \) into a Coxeter group with generators given by reflections with respect to simple roots.

Then \( \phi \) sends \( w \) to \( \sum_{y \leq w} n_{w,y} T_y \) with \( n_{w,w} = 1 \). An immediate corollary is as follows.

Corollary 4.3. \( H_m(Z) \) is semisimple.

Proof. Every group algebra of a finite group over a field whose characteristic does not divide the order of the group is semisimple. (cf. unitary trick)

4.2 Springer Correspondence

In this section, we deal with the natural representations of \( H_m(Z) \) given by geometrical construction, similar to Proposition 1.4. Let \( \mu : \tilde{\mathcal{N}} \to \mathcal{N} \) be the Springer resolution and \( Z = \tilde{\mathcal{N}} \times_{N} \tilde{\mathcal{N}} \) be the Steinberg variety. Then for \( x \in \mathcal{N} \), \( Z_x = B_x \times B_x \), and since \( Z \circ Z_x = Z_x = Z_x \circ Z \), \( H_{2d}(Z_x) \) is equipped with the natural \( H_m(Z) \)-bimodule structure where \( d = \dim_{\mathbb{R}} B_x \). Furthermore, if we investigate this construction, and by Kunneth isomorphism, we obtain a following lemma.

Lemma 4.4. \( H_{2d}(Z_x) = H_d(B_x) \otimes H_d(B_x) \) as \( H_m(Z) \)-bimodules.

Also, we know that \( C_x = G_x/G^0_x \) acts on \( H_d(B_x) \), since any action of a connected group becomes trivial on the homology level. Then we may consider the relation between this and \( H_m(Z) \)-action, which is as follows.

Lemma 4.5. (Left or right) \( H_m(Z) \)-action on \( H_d(B_x) \) is compatible with the morphism \( g : H_d(B_x) \to H_d(B_{gxg^{-1}}) \) derived from the adjoint action for all \( g \in G \), thus it commutes with the natural \( C_x \)-action on \( H_d(B_x) \).

Thus we can decompose \( H_m(B_x) \) (over \( \mathbb{C} \)) into

\[
\mathbb{C} \otimes_{\mathbb{Q}} H_d(B_x) = \bigoplus_{\chi \in C^\wedge_x} \mathbb{C} \otimes_{\mathbb{Q}} H_d(B_x)_\chi
\]

as left \( H_m(Z) \)-modules, where \( C^\wedge_x \) is the set of all irreducible representations that occur in \( H_d(B_x) \). Now since the adjoint action of \( g \in G \) on \( \mathcal{N} \) gives an isomorphism between \( H_d(B_x) \) and \( H_d(B_{gxg^{-1}}) \), we obtain

\[
\bigoplus_{\chi \in C^\wedge_x} \mathbb{C} \otimes_{\mathbb{Q}} H_d(B_x)_\chi = \bigoplus_{\phi \in C^\wedge_{gxg^{-1}}} \mathbb{C} \otimes_{\mathbb{Q}} H_d(B_{gxg^{-1}})_\phi
\]

and \( g \in G \) gives the bijections \( C^\wedge_x \to C^\wedge_{gxg^{-1}} \). Now we state the following theorem, which is our ultimate goal of this article.
Theorem 4.6. There exists a following bijection.

\[ \{(x, \chi) | \chi \in C^n \}/G \longleftrightarrow \{\text{Complex irreducible representations of } H_m(Z)\} \]

\[ (x, \chi) \longleftrightarrow H_d(B_x) \chi \quad (d = \dim \mathcal{B}_x) \]

In other words, we can label all complex irreducible representations of \( H_m(Z) \) uniquely by orbits of the form \((x, \chi)\).

5 Example: \( \mathfrak{sl}_n \)

In this section, we give the clearest and the most fundamental example, namely the Springer correspondence for \( \mathfrak{sl}_n \). The speaker may delve into this example after finishing all the explanation above, or draw a vertical line at the center of a blackboard and write the mirror image in the case of \( \mathfrak{sl}_n \) in one side while describe the general frame on the other side.

In this section, \( G = SL_n \) and \( \mathfrak{g} = \mathfrak{sl}_n \). (All the objects are over \( \mathbb{C} \).) Also we fix \((B, T)\) as the set of upper triangular matrices and diagonal matrices. Then \((\mathfrak{b}, \mathfrak{h})\) corresponds to upper triangular matrices and diagonal matrices (in \( \mathfrak{g} \)). The set of simple roots are given by \( \{e_{i,i} - e_{i+1,i+1}\}_{1 \leq i \leq n-1} \), and the Weyl group is \( S_n \), since \( N(T)/T \) can be identified with the set of permutation matrices.

First of all, \( G/B \simeq B \) is equivalent to the complete flag variety of \( \mathbb{C}^n \). Indeed, for any complete flag \( F \) one can associate this to the Borel subalgebra of \( \mathfrak{g} \), say the set of "upper-triangular" matrices with respect to the "coordinates" induced by \( F \). Since this mapping has an inverse, it is injective, and it is also surjective by Lie’s theorem. (\( \mathfrak{b} \) is solvable.)

Clearly \( W_T \) acts on \( \mathfrak{h} \) by permutation of diagonal terms, and thus an element of \( \mathfrak{h} \) is regular if and only if all the diagonal entries are different form one another. Also, in this case there exist exactly \( n! \) Borel subalgebras, since they are equivalent to ordering \( n \) different eigenspaces which are distinguishable from one another.

Now we investigate \( \mathcal{N}^{reg} \). The only regular nilpotent matrix in Jordan canonical form consists of a single block, and thus \( \mathcal{N}^{reg} \) is just its conjugates. In other words, it is the single orbit. Also, there exists only one Borel subalgebra which contains such nilpotent matrix, which is the same as our previous choice. Thus \( \tilde{\mathcal{N}}^{reg} = \mathcal{N}^{reg} \).

Meanwhile, Chevalley’s theorem, i.e., \( \mathbb{C}[\mathfrak{g}]/G = \mathbb{C}[\mathfrak{h}]^{W} \), is in this case obvious, since both are the set of symmetric polynomials of eigenvalues, i.e., coefficients of characteristic polynomials. Now with this data one can reformulate Grothendieck and Springer resolution for \( \mathfrak{sl}_n \).

Now we deal with the Springer correspondence of this algebra. Since we deal with orbits in \( \mathcal{N} \), we only need to consider matrices of Jordan canonical form with zero diagonal entries. Also, two such matrices are adjoint if and only if they consist of the same size of blocks, thus we have a bijection

\[ \{\text{orbits in } \mathcal{N}\} \longleftrightarrow \{\text{partitions of } n\} \]
The main obstacle for this argument is that $C_x$ can be nontrivial. However, we circumvent this phenomenon by considering $GL_n$ rather than $SL_n$. At a glance it seems unreasonable since $GL_n$ is not semisimple, but in fact all the argument above can also be justified when the Lie algebra is reductive, and furthermore, the whole theory is identical when the reductive algebra $\mathfrak{g}$ is replaced with $[\mathfrak{g},\mathfrak{g}]$. ($\mathcal{B}, \mathcal{N}$, and $Z$ are not changed.) Then we may use the following lemma.

**Lemma 5.1.** $G_x \subset GL_n$ is connected for all $x \in \mathfrak{gl}_n$.

**Proof.** The condition $gx = xg$ for $g \in M_n$ defines a vector subspace of $M_n$. The condition $\det = 0$ is codimension 1, which is real codimension 2 in that vector subspace. \(\square\)

Or, we may just use this lemma to show that in fact $C_x \subset SL_n$ acts trivially on $H_d(\mathcal{B}_x)$ (whether or not $C_x$ is trivial.) Then the Springer correspondence tells that orbits in $\mathcal{N}$ parametrize all the irreducible representations of $S_n$. In other words, we have the following bijections

\[
\{\text{irreducible representations of } S_n\} \leftrightarrow \{\text{orbits in } \mathcal{N}\} \leftrightarrow \{\text{partitions of } n\}
\]

Also, we have the following identity

\[
\#S_n = \#\{\text{components of } Z\} = \sum_{\mathbb{O}} \#\{\text{components of } Z_\mathbb{O}\} = \sum_{\mathbb{O}} \#\{\text{components of } \mathcal{B}_x \text{ for some (any) } x \in \mathbb{O}\}^2
\]

\[
(\because Z_\mathbb{O} \simeq \cup_{i,j} \mathcal{B}_x^i \times \mathcal{B}_x^j \text{ for some } x \in \mathbb{O})
\]

\[
= \sum_{\mathbb{O}} (\dim H_{d(x)}(\mathcal{B}_x))^2 = \sum_{V \in \hat{S}_n} (\dim V)^2
\]

Also, if we count the number of fixed components by the involution of switching two factors we obtain (in fact, first we need to prove that $w \mapsto w^{-1}$ on $\mathbb{W}$ corresponds to switching factors.)

\[
\#\{\text{involutions in } S_n\} = \sum_{\mathbb{O}} \#\{\text{components of } \mathcal{B}_x \text{ for some (any) } x \in \mathbb{O}\}
\]

\[
= \sum_{\mathbb{O}} (\dim H_{d(x)}(\mathcal{B}_x)) = \sum_{V \in \hat{S}_n} (\dim V)
\]
6 Further Stories

In this section, we will adopt another approach to investigate the structure of $H_m(Z)$, which is roughly a sheaf-theoretic point of view. Almost impossible is to prove all the arguments in this section, we basically omit and just admit steps to be used further, but try to sketch or at least catch a glimpse of what is really going on here. For preliminaries, it will be quite helpful to have understood (or have some knowledge of) the theory of derived categories, local systems and monodromy representations, and perverse sheaves. In particular, some arguments in this section would be straightforwardly understood if used to the structure of the category of perverse sheaves and the decomposition theorem of Beilinson, Bernstein, and Deligne. We will briefly introduce requisite arguments, but do not delve into each subject in detail.

6.1 Stratification and Constructability

Throughout this section, $X$ is a complex algebraic variety. Let $\text{Sh}(X)$ denote the category of sheaves of complex vector spaces on $X$, and $\text{LS}(X)$ the full subcategory of local systems (locally constant sheaves of finite-dimensional complex vector spaces.)

Definition 6.1. A finite partition $X = \sqcup_i X_i$ is an algebraic stratification of $X$ if

- each piece $X_i$ is a smooth locally closed algebraic subvariety of $X$,
- for any $j \in I$ the closure of $X_j$ is a union of some $X_i$’s, and
- for any $j \in I$ and any $y \in X_j$ there exists a stratified slice to $X_j$ at $y$.

Here, a locally closed (w.r.t. ord. top.) complex analytic subset $S \subset X$ is a transverse slice to $X_j$ at $y$ if $y \in S$ and there exists an open (w.r.t. ord. top.) $y \in U \subset X$ and an analytic isomorphism $f : (Y \cap U) \times S \simeq U$ such that $f_{|\{y\} \times S} \simeq id_S$ and $f_{|(Y\cap U) \times \{y\}} \simeq id_{Y\cap U}$. It is called a stratified slice if this isomorphism preserves the partitions $(X_j \cap U) \times S = \sqcup_i ((X_j \cap U) \times (S \cap X_i))$ and $U = \sqcup_i (X_i \cap U)$. In other words, it derives isomorphisms $(X_j \cap U) \times (S \cap X_i) \simeq X_i \cap U$ for all $i \in I$.

A typical example is as follows.

Proposition 6.2. Let $V$ be a smooth algebraic $G$-variety and $X \in V$ a $G$-stable algebraic subvariety consisting of finitely many $G$-orbits. Then the partition of $X$ into $G$-orbits is an algebraic stratification of $X$.

Definition 6.3. $\mathcal{F} \in \text{Sh}(X)$ is constructible if there exists an algebraic stratification $X = \sqcup_i X_i$ such that $\mathcal{F}|_{X_i} \in \text{LS}(X_i)$ for all $i$. We let $D^b(X)$ be a full subcategory of $D^b(\text{Sh}(X))$ consisting of all the objects whose cohomology sheaves are constructible.

However, differently than one might suspect, $D^b(X)$ is not the derived category of the category of constructible sheaves.
6.2 Verdier Duality and Grothendieck’s Six Operations

Use the same notation as before. Now fix a closed embedding \(i: X \hookrightarrow M\) to a smooth manifold \(M\) (it always exists.) Then we define \(\Gamma_X(V, F) \subset \Gamma(V, F)\) as a subgroup consisting of all sections whose support is contained in \(X\). Then one can check that the presheaf \(V \mapsto \Gamma_X(V, F)\) is actually a sheaf, and may consider it as a sheaf on \(X\). Now we define a functor \(i_!\): \(\text{Sh}(M) \to \text{Sh}(X)\) which gives the corresponding sheaf on \(X\). Equivalently, \(i_!(\mathcal{F})(V) = \varprojlim_{V \subset U} \Gamma_X(U, \mathcal{F})\).

Then one can directly show that \(i_!\) is left exact, thus we have a derived functor \(Ri_!\): \(D^b(\text{Sh}(M)) \to D^b(\text{Sh}(X))\). Furthermore, this functors sends \(D^b(M)\) to \(D^b(X)\) if \(M\) is also an algebraic variety.

**Definition 6.4.** Let \(\mathbb{C}_M \in D^b(M)\) be the constant sheaf. We call \(D_X = Ri_!(\mathbb{C}_M)[2 \dim \mathbb{C}_M]\) the dualizing complex of \(X\).

Meanwhile, the hypercohomology of the dualizing complex, namely \(H^*(X, \mathbb{D}_X)\), is by definition applying the derived functor of the global section functor on \(X\) to \(\mathbb{D}_X\), thus \(H^*(X, \mathbb{D}_X) = R^*\Gamma(X, \mathbb{D}_X) = R^*\Gamma(X, Ri_!\mathbb{C}_M[2 \dim \mathbb{C}_M]) = R(\Gamma \circ i_!)(\mathbb{C}_M[2 \dim \mathbb{C}_M]) = R^{\dim \mathbb{C}_M} \Gamma_X(\mathbb{C}_M).\) However it is nothing but the relative cohomology \(H^{\dim \mathbb{C}_M}(M, M \setminus X)\), and by Poincaré duality it equals \(H^{\dim \mathbb{C}_M}(X)\), i.e. the Borel-Moore homology group. Also it has a local counterpart; for \(x \in X\), \((H^*(\mathbb{D}_X))_x = H^{\dim \mathbb{C}_M}(U \cap X)\), where \(x \in U\) is a small contractible neighborhood.

Indeed, the dualizing complex \(\mathbb{D}_X\) is intrinsic. Precisely, we have

**Lemma 6.5.** If \(i: N \hookrightarrow M\) is a closed embedding between smooth complex varieties, then \(Ri_!(\mathbb{C}_M) = \mathbb{C}_N[2(\dim \mathbb{C}_N - \dim \mathbb{C}_M)].\) That is, the two definition of \(\mathbb{D}_N\) computed by \(i\) and \(id_N\) coincide. Also, in general, \(\mathbb{D}_X\) does not depend on the choice of an embedding \(i: X \hookrightarrow M\) regardless of its smoothness.

Now, we define the Verdier duality.

**Definition 6.6.** The Verdier duality functor is defined by \(\mathbb{A}^{-} : D^b(X)^{\text{op}} \to D^b(X) : A \mapsto \mathcal{H}om(A, \mathbb{D}_X)\) where \(\mathcal{H}om = R\mathcal{H}om_{\text{Sh}(X)}\).

**Proposition 6.7.** (a) \((\mathbb{A}^{-})^{-} = A\).
(b) \((\mathcal{F}[n])^{-} = (\mathcal{F}^{-})[-n].\)
(c) \(\mathbb{C}_X^{-} = \mathbb{D}_X.\)

Now we define the following functors. They are often called (excluding \(\otimes^i\) and including \(\mathcal{H}om\)) Grothendieck’s six operations.

**Definition 6.8.** For \(f: X \to Y\), we have the following functors on \(D^b\).

- The pullback functor \(f^* = f^{-1}\). Being already exact, it is also well-defined in \(D^b\).
• The pushforward functor $Rf_*$ is the derived functor of the usual pushforward functor. We abuse the notation and denote it by $f_*$.

• The extraordinary (or proper) pushforward functor $f! = \vee f_* \vee$.

• The extraordinary (or proper) pullback functor $f^! = \vee f^* \vee$.

• The tensor product $\otimes = \Delta^*(- \boxtimes -)$ and $\otimes^! = \Delta^!(- \boxtimes -)$ where $\boxtimes$ is the external tensor product.

**Proposition 6.9.** (a) $(A \otimes B)^\vee = A^\vee \otimes^! B^\vee$ and $(A \otimes^! B)^\vee = A^\vee \otimes B^\vee$, since $\vee$ commutes with $\boxtimes$.

(b) $f^* \dashv f_*$, and $f_! \dashv f^!$.

(c) $H^*(Y, f_* A) = H^*(X, A)$, and $H^*_c(Y, f! A) = H^*_c(X, A)$.

(d) $f^! C_Y = C_X$, and $f^! D_Y = D_X$.

(e) If $f$ is proper, then $f_! = f_*$.

(f) If $f$ is flat with smooth fibers of complex dimension $d$, i.e., $f$ is smooth with relative dimension $d$, then $f^! = f^*[2d]$.

(g) If $f$ is a closed embedding, then the two definitions of $f^!$ (as derived functors) coincide.

(h) **(Base change)** If the following diagram is a cartesian square,

\[
\begin{array}{ccc}
X \times_{\overline{Z}} Y & \xrightarrow{f} & Y \\
\downarrow{\bar{g}} & & \downarrow{g} \\
X & \xrightarrow{f} & Z
\end{array}
\]

then $g^! f_* = \bar{f}_* \bar{g}^!$.

(i) $A \otimes C_X = A \otimes^! D_X = A$.

(j) $\mathcal{H}om(A, B) = \mathcal{H}om(A, \mathcal{H}om(B^\vee, D_X)) = \mathcal{H}om(A \otimes B^\vee, D_X) = A^\vee \otimes^! B$.

(k) $Ext^*(A, B) = H^*(X, \mathcal{H}om(A, B)) = H^*(X, A^\vee \otimes^! B) = Ext^*(C_X, A^\vee \otimes^! B)$.

With these properties, one can reformulate some properties of Borel-Moore homology using the fact that $D_X$ computes those groups.

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6.3 Sheaf-Theoretic Analysis of the Convolution Algebra

We now define the constant perverse sheaf.

**Definition 6.10.** For a smooth variety $X$ with irreducible components $X_i$’s, we define the constant perverse sheaf on $X$, denoted by $\mathcal{C}_X \in D^b(X)$, characterized by $\mathcal{C}_X|_{X_i} = \mathbb{C}_{X_i}[\dim_{\mathbb{C}} X_i]$.

**Proposition 6.11.** It is well-defined, and self-dual; i.e., $\mathcal{C}_X^\vee = \mathcal{C}_X$.

Now we assume to be given a smooth complex variety $X$ and a proper map $\mu : X \to Y$ between complex varieties. We set $Z = X \times_Y X$. A typical example is our Steinberg variety $Z = \tilde{N} \times_{\mathcal{N}} \tilde{N}$. In this case, $\mathcal{C}_X = \mathcal{C}_X[\dim_{\mathbb{C}} X]$. The main theorem in this section is the following.

**Theorem 6.12.** There is a natural (not grading-preserving) algebra isomorphism

$$H_*(Z) = \text{Ext}^{2 \dim_{\mathbb{C}} X - \bullet}(\mu_* \mathcal{C}_X, \mu_* \mathcal{C}_X).$$

**Sketch of proof.** We only prove that this is a vector space isomorphism. Let $p_1, p_2 : Z \to X$ be natural projections. Then we have

$$\text{Ext}^*(\mu_* \mathcal{C}_X, \mu_* \mathcal{C}_X) = H^*(Y, \text{Hom}(\mu_* \mathcal{C}_X, \mu_* \mathcal{C}_X))$$

$$= H^*(Y, (\mu_* \mathcal{C}_X)^\vee \otimes^1 (\mu_* \mathcal{C}_X))$$

$$= H^*(Y, \Delta^1((\mu_* \mathcal{C}_X)^\vee \otimes (\mu_* \mathcal{C}_X)))$$

where $\Delta : Y \hookrightarrow Y \times Y$

$$= H^*(Y, \Delta^1((\mu_! \mathcal{C}_X^\vee) \boxtimes (\mu_* \mathcal{C}_X)))$$

since $\mu$ is proper

$$= H^*(Y, \Delta^1(\mu \times \mu)_*(\mathcal{C}_X^\vee \boxtimes \mathcal{C}_X))$$

$$= H^*(Y, (\mu \times_Y \mu)_! (\mathcal{C}_X^\vee \boxtimes \mathcal{C}_X))$$

by base change, where $i : Z \hookrightarrow X \times X$

$$= H^*(Z, i_! (\mathcal{C}_X^\vee \boxtimes \mathcal{C}_X))$$

$$= H^*(Z, i_! (\mathcal{C}_X \times_X [2 \dim_{\mathbb{C}} X]))$$

$$= H^*(Z, i_! (\mathbb{D}_X \times_X [-2 \dim_{\mathbb{C}} X]))$$

by Poincaré duality.

$$= H^*_{2 \dim_{\mathbb{C}} X - \bullet}(Z)$$

Considering the zeroth degree part on the LHS, we obtain the following corollary.

**Corollary 6.13.** For $m = \dim_{\mathbb{R}} X = 2 \dim_{\mathbb{C}} X$, we have $H_m(Z) = \text{End}(\mu_* \mathcal{C}_X)$.
6.4 Local Systems and Monodromy Representation

Before we delve into the subject of perverse sheaves, we briefly check the correspondence between local systems and monodromy representations. One can show the following result.

**Theorem 6.14.** Let $X$ be a "nice" connected space. For $x \in X$, there exists an equivalence of categories

$$\text{Loc}(X) \leftrightarrow \text{Rep}(\pi_1(X, x))$$

$$L \mapsto \pi_1(X, x) \curvearrowright L$$

$$\tilde{X} \times V/\pi_1(X, x) \to X \leftarrow V$$

where $\tilde{X}$ is the universal covering of $X$.

6.5 Perverse Sheaves

Thanks to prior lectures, we take the theory of perverse sheaves for granted. Also, we use the following notation.

**Definition 6.15.** For an open dense subset $U \subset X^{sm}$, $i : U \to X$, and a local system $L \in \text{Loc}(U)$, we define $\text{IC}_X(L) = \text{im}(p_i! L[\dim_C X] \to p_i^* L[\dim_C X]) \in D^b(X)$.

**Proposition 6.16.** $\mathcal{C}_X$ is well-defined and is a perverse sheaf on $X$. Indeed, $\mathcal{C}_X = \text{IC}_X(C_{X^{sm}})$ where $i : X^{sm} \to X$ is a smooth dense open subset of $X$.

We now state the decomposition theorem of Beilinson-Bernstein-Deligne. For further discussion, the readers are referred to [HTT].

**Theorem 6.17** (Decomposition theorem, BBD). For a proper morphism $f : X \to Y$ of algebraic varieties, we have

$$Rf_*[\mathcal{C}_X] \simeq \bigoplus_k i_k^* \text{IC}_{Y_k}(L_k)[l_k],$$

where $Y_k$ is an irreducible closed subvariety of $Y$, $i_k : Y_k \to Y$ is the corresponding embedding, $L_k \in \text{Loc}(U_k)$ is a local system on $U_k \subset Y_k^{sm}$, and $l_k \in \mathbb{Z}$.

In our case, this theorem can be elaborated further. Let us first define some relevant notions.

**Definition 6.18.** Let $f : X \to Y$ be a dominant morphism of irreducible algebraic varieties. Then $f$ is small (resp. semismall) if $\text{codim}_Y \{ y \in Y \mid \dim f^{-1}(y) \geq k \} > 2k$ (resp. $\geq 2k$) for all $k \geq 1$. 

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**Definition 6.19.** Let $X$ be an algebraic variety of an analytic space. $X$ is called rationally smooth if for any $x \in X$, $H^i_{\{x\}}(X, \mathbb{C}_X) = \mathbb{C}$ if $i = 2 \dim X$ and 0 otherwise.

Obviously, smooth means rationally smooth, thus $\tilde{N}$ is rationally smooth. Furthermore, we have

**Theorem 6.20.** $\tilde{N} \to N$ is semismall.

*Proof.* We decompose $\mu : \tilde{N} \to N$ into $G$-orbits, i.e., $\mu_\mathcal{O} = \mu|_{\mu^{-1}\mathcal{O}} : \mu^{-1}\mathcal{O} \to \mathcal{O}$. Each of these is a fiber bundle by definition of $\mu$. Then by Theorem 3.16, we have $\dim \mathcal{O} + 2 \dim \mathcal{B}_x = 2 \dim \mathcal{B} = \dim N = \dim \tilde{N}$ for $x \in \mathcal{O}$. It implies for every $k \geq 1$,

$$\{ x \in N | \dim \mu^{-1}(x) \geq k \} = \bigsqcup_{\text{codim}_N \mathcal{O} \geq 2k} \mathcal{O},$$

from which the theorem follows. $\square$

In general, for a dominant projective morphism $f : X \to Y$ of irreducible algebraic varieties, there exists a complex stratification $Y = \sqcup Y_\alpha$ such that $Y_\alpha$ is connected and on each stratum $f$ is topologically a fiber bundle. In this case, $Y_\alpha$ is called relevant if $\text{codim}_Y Y_\alpha = 2 \dim F_\alpha$, where $F_\alpha$ is the corresponding fiber over $Y_\alpha$. Then Theorem 3.16 can be rephrased as the following

**Theorem 6.21.** Every $G$-orbit in $N$ is relevant.

Now, as promised before, we state a more elaborate form of the decomposition theorem.

**Theorem 6.22.** (Borho-MacPherson). Let $f : X \to Y$ be semismall, and $\sqcup Y_\alpha$ is such a stratification explained above. Also assume $X$ is rationally smooth. Then we have

$$Rf_*[\mathbb{C}_X] \simeq \bigoplus_{Y_\alpha, \phi} (i_\alpha*IC_{Y_\alpha}(L_\phi))^\boxtimes m_\phi \simeq \bigoplus_{Y_\alpha, \phi} i_\alpha*IC_{Y_\alpha}(L_\phi) \otimes V_{Y_\alpha, \phi},$$

where $(Y_\alpha, \phi)$ is a pair of a stratum and an irreducible representation of $\pi_1(Y_\alpha)$, $m_\phi$ is the multiplicity of the local system $L_\phi$ corresponding to $\phi$, and $i_\alpha : Y_\alpha \subseteq Y$ is the embedding. Furthermore, $V_{Y_\alpha, \phi}$ is determined by

$$H^{\dim_x B_x}(B_x) \simeq \bigoplus_{\phi} \phi \otimes V_{Y_\alpha, \phi}$$

for $x \in Y_\alpha$.

Thus, we have the following theorem, which is our final goal.

**Theorem 6.23.** $H^{\dim N}(Z) = \bigoplus_{Y_\alpha, \phi} \text{End}(V_{Y_\alpha, \phi})$. Thus, $H^{\dim N}(Z)$ is semisimple.
It is the direct consequence of the following proposition.

**Proposition 6.24.** Let $\phi, \psi$ be irreducible representations of $\pi_1(Y_\alpha), \pi_1(Y_\beta)$, respectively. Then $\text{Hom}_{D^b(X)}(i_{\alpha*}(L_\phi), i_{\beta*}(L_\psi)) = \mathbb{C} \cdot \delta_{\alpha, \beta} \delta_{\phi, \psi}$.

Lastly, we give some useful simple observations. Let $G$ be a Lie group and $H$ be a Lie subgroup of $G$ such that $H \to G \to G/H$ is a (Serre) fibration. Then we obtain a following exact sequence:

$$
\cdots \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1.
$$

Thus, for example, if $G = GL_n(\mathbb{C})$, then it is simply connected, thus $\pi_1(G/H) \cong \pi_0(H) = H/H^0$. As a result, every nilpotent orbit in $\mathfrak{gl}_n(\mathbb{C})$ has a trivial fundamental group. Using this, one can also argue that the action of $\pi_1(O)$ for a nilpotent orbit $O$ of $\mathfrak{sl}_n$ on $H^{dim B_x}(B_x)$ is trivial. (Note that the group is not trivial per se; see [CM]_.) Furthermore, it shows that the two explanations of the convolution algebra actually nearly coincide with each other.

**References**


