1. Previously in the Seminar...

We recall some preliminaries discussed in previous talks. We always assume that \(X\) is a complex manifold.

**Definition 1.1.** A filtered regular holonomic \(\mathcal{D}\)-module with \(\mathbb{Q}\)-structure is a triple \(\mathcal{M} = (\mathcal{M}, F_{\bullet} \mathcal{M}, K)\) consisting of the following objects:

1. A constructible complex of \(\mathbb{Q}\)-vector spaces \(K\).
2. A regular holonomic right \(\mathcal{D}_X\)-module \(\mathcal{M}\) with an isomorphism
   \[
   DR(\mathcal{M}) \simeq \mathbb{C} \otimes_{\mathbb{Q}} K.
   \]
   By the Riemann-Hilbert correspondence, this makes \(K\) a perverse sheaf.
3. A good filtration \(F_{\bullet} \mathcal{M}\) by \(\mathcal{O}_X\)-coherent subsheaves of \(\mathcal{M}\), such that
   \[F_p \mathcal{M} \cdot F_k \mathcal{D} \subset F_{p+k} \mathcal{M}\]
   and such that \(gr^F \mathcal{M}\) is coherent over \(gr^F \mathcal{D}_X \simeq \text{Sym}^\bullet T_X\).

Its Tate twist is defined by \(M(k) = (\mathcal{M}, F_{\bullet-k} \mathcal{M}, K \otimes \mathbb{Q} Q(k))\) where \(Q(k) = (2\pi i)^k \mathbb{Q} \subset \mathbb{C}\).

For a given function \(f : X \to \mathbb{C}\), we want to define the nearby and vanishing cycles, denoted \(\psi_f\) and \(\phi_f\), in the category of filtered regular holonomic \(\mathcal{D}\)-modules with \(\mathbb{Q}\)-structure. If \(f^{-1}(0)\) is not smooth, instead of \(\mathcal{M} = (\mathcal{M}, F_{\bullet} \mathcal{M}, K)\) we consider \((id, f)_* \mathcal{M} = (\mathcal{M}_f, F_{\bullet} \mathcal{M}_f, K_f)\) where

\[
\mathcal{M}_f = (id, f)_+ \mathcal{M} = \mathcal{M}[\partial_t], \quad F_{\bullet} \mathcal{M}_f = F_{\bullet}(id, f)_+ \mathcal{M} = \bigoplus_{i=0}^{\infty} F_{\bullet-i} \mathcal{M} \otimes \partial_t^i, \quad K_f = f_* K.
\]
Now we recall the definition of nearby and vanishing cycles of $\mathcal{D}$-modules. To that end we will only consider objects with quasi-unipotent local monodromy, thus the eigenvalues of monodromy operator $T$ on $\mathcal{M}_f$ are rational. Then we defined

$$
\psi_f \mathcal{M}_f = \bigoplus_{-1 \leq \alpha < 0} \text{gr}^V_{\alpha} \mathcal{M}_f \simeq \bigoplus_{-1 \leq \alpha < 0} \mathcal{M}_{f,\alpha}, \quad \phi_f \mathcal{M}_f = \bigoplus_{-1 < \alpha \leq 0} \text{gr}^V_{\alpha} \mathcal{M}_f \simeq \bigoplus_{-1 < \alpha \leq 0} \mathcal{M}_{f,\alpha}
$$

where the superscript $V$ denotes the (rational) $V$-filtration, which is a refinement of the Kashiwara-Malgrange filtration. Note that these $\mathcal{D}$-modules are regular holonomic if $\mathcal{M}_f$ is regular holonomic.

We recall some properties of $V$-filtration.

(1) $t \partial_t - \alpha$ acts nilpotently on $\text{gr}^V_{\alpha} \mathcal{M}_f$.

(2) $t : \text{gr}^V_{\alpha} \mathcal{M}_f \rightarrow \text{gr}^V_{\alpha - 1} \mathcal{M}_f$ is an isomorphism for $\alpha < 0$.

(3) $\partial_t : \text{gr}^V_{\alpha} \mathcal{M}_f \rightarrow \text{gr}^V_{\alpha + 1} \mathcal{M}_f$ is an isomorphism for $\alpha \neq -1$.

(4) $V_{<0} \mathcal{M}_f$ only depends on the restriction of $\mathcal{M}$ to $X - X_0$.

2. NEARBY AND VANISHING CYCLES

On the perverse sheaf side, we also have $p\psi_f K, p\phi_f K,$ and

$$
p\psi_f K := \psi_f K[-1], \quad p\phi_f K := \phi_f K[-1]
$$

are perverse sheaves by Gabber’s theorem. As the category of perverse sheaves is abelian we have

$$
p\psi_f K = \bigoplus_{\lambda \in \mathbb{C}^*} p\psi_{f,\lambda} K, \quad p\phi_f K = \bigoplus_{\lambda \in \mathbb{C}^*} p\phi_{f,\lambda} K
$$

where $\psi_{f,\lambda} K, \phi_{f,\lambda} K$ are generalized eigenspaces of $T$ corresponding to eigenvalue $\lambda$.

These two notions of nearby and vanishing cycles are connected by de Rham functor as follows. (Here $\lambda = e^{2\pi i \alpha}$.)

$$
\text{DR}(\mathcal{M}_{f,\alpha}) \simeq p\psi_{f,\lambda}(\text{DR}(\mathcal{M})) \text{ for } -1 \leq \Re \alpha < 0 \quad \text{DR}(\mathcal{M}_{f,\alpha}) \simeq p\phi_{f,\lambda}(\text{DR}(\mathcal{M})) \text{ for } -1 < \Re \alpha \leq 0
$$

Thus we have

$$
\text{DR}(\psi_f \mathcal{M}_f) \simeq \mathbb{C} \otimes_{\mathbb{Q}} p\psi_f K, \quad \text{DR}(\phi_f \mathcal{M}_f) \simeq \mathbb{C} \otimes_{\mathbb{Q}} p\phi_f K.
$$

This suggests that we should have

$$
\psi_f M = \left( \bigoplus_{-1 \leq \alpha < 0} \text{gr}^V_{\alpha} \mathcal{M}_f, ???, p\psi_f K \right),
$$

$$
\phi_f M = \left( \bigoplus_{-1 < \alpha \leq 0} \text{gr}^V_{\alpha} \mathcal{M}_f, ???, p\phi_f K \right),
$$

$$
\psi_{f,\lambda} M = \left( \text{gr}^V_{\alpha} \mathcal{M}_f, ???, p\psi_{f,\lambda} K \right),
$$

$$
\phi_{f,\lambda} M = \left( \text{gr}^V_{\alpha} \mathcal{M}_f, ???, p\phi_{f,\lambda} K \right)
$$
where $\lambda = e^{2\pi i \alpha}$. But some of them are not well-defined, because:

1. For $1 \neq \lambda \in \mathbb{C}^\times$ in general $p\psi_{f,\lambda}K \simeq p\phi_{f,\lambda}K$ is not defined over $\mathbb{Q}$. But $p\psi_fK, p\phi_fK, p\psi_{f,1}K, p\phi_{f,1}K$ are defined over $\mathbb{Q}$.

2. If we want to define $\partial_t : \psi_fM \to \phi_fM$ and $t : \phi_fM \to \psi_fM(-1)$, we are tempted to adopt the following definition

$$\phi_fM = \left( \bigoplus_{0 \leq \alpha < 1} \text{gr}_\alpha^V M_f, ???, p\phi_fK \right)$$

instead of the above. But $\text{DR}(\bigoplus_{0 \leq \alpha < 1} \text{gr}_\alpha^V M_f) \not\simeq p\phi_fK$. (We already observed that $\text{DR}(\bigoplus_{-1 < \alpha \leq 0} \text{gr}_\alpha^V M_f) \not\simeq p\phi_fK$.) Thus in order to define $\text{var}$ and $\text{can}$, we need (and it suffices) only to define $\phi_{f,1}M$.

Thus we will define

$$\psi_fM = \left( \bigoplus_{-1 \leq \alpha < 0} \text{gr}_\alpha^V M_f, ???, p\psi_fK \right),$$

$$\psi_{f,1}M = (\text{gr}_{-1}^V M_f, ???, p\psi_{f,1}K),$$

$$\phi_{f,1}M = (\text{gr}_0^V M_f, ???, p\phi_{f,1}K)$$

It remains to define the filtration on each $\mathcal{D}$-module. For each $\text{gr}_\alpha^V M_f$, there is only one reasonable way to define such filtration; we define

$$F_p \text{gr}_\alpha^V M_f := F_p M_f \cap V_\alpha M_f / F_p M_f \cap V_{<\alpha} M_f$$

which is the induced filtration. On the other hand, the filtration on $\bigoplus_{-1 \leq \alpha < 0} \text{gr}_\alpha^V M_f$ is a bit subtle; there are two ways to define "induced filtrations" on it.

1. $F_p \left( \bigoplus_{-1 \leq \alpha < 0} \text{gr}_\alpha^V M_f \right) := F_p M_f \cap V_{<0} M_f / F_p M_f \cap V_{<1} M_f$

2. $F_p \left( \bigoplus_{-1 \leq \alpha < 0} \text{gr}_\alpha^V M_f \right) := \bigoplus_{-1 \leq \alpha < 0} F_p M_{f,\alpha}$

Here we adopt the second definition, which "respects" each graded piece of $\bigoplus_{-1 \leq \alpha < 0} \text{gr}_\alpha^V M_f$. It turns out to be a good notion...

To summarize, in the category of filtered regular holonomic $\mathcal{D}$-modules with $\mathbb{Q}$-structure we may define

$$\psi_fM = \bigoplus_{-1 \leq \alpha < 0} (\text{gr}_\alpha^V M_f, F_{-1} \text{gr}_\alpha^V M_f, p\psi_{f,e^{2\pi i \alpha}} K)$$

$$\psi_{f,1}M = (\text{gr}_{-1}^V M_f, F_{-1} \text{gr}_{-1}^V M_f, p\psi_{f,1}K)$$

$$\phi_{f,1}M = (\text{gr}_0^V M_f, F_0 \text{gr}_0^V M_f, p\phi_{f,1}K)$$

provided the filtrations are good. Also $\partial_t, t$ give the following maps

$$\text{can} : \psi_{f,1}M \to \phi_{f,1}M, \quad \text{var} : \phi_{f,1}M \to \psi_{f,1}M(-1).$$
The compositions can $\circ \varnothing, \varnothing \circ$ can are the same as $N = \frac{1}{2\pi i} \log T_n$. As it is nilpotent, we have a natural filtration induced by $N$, called monodromy weight filtration, denoted $W_*\psi_f M$ and $W_*\phi_{f,1} M$.

3. STRICT SUPPORT

**Definition 3.1.** Let $Z \subset X$ be an irreducible subvariety. We say that a $D$-module $\mathcal{M}$ has strict support $Z$ if the support of every nonzero subobject or quotient object of $\mathcal{M}$ is equal to $Z$.

For a regular holonomic $\mathcal{M}$, it has strict support $Z$ if and only if $DR(\mathcal{M})$ is an intersection cohomology complex of a local system on a Zariski open subset of $Z$. Also we have the following.

**Lemma 3.2.** Let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function and $\mathcal{M}$ be a regular holonomic $D$-module on $X$. Then

1. $\mathcal{M}$ has no nonzero subobject supported on $f^{-1}(0) \iff t : \text{gr}_0^V M_f \to \text{gr}_1^V M_f$ is injective.

2. $\mathcal{M}$ has no nonzero quotient object supported on $f^{-1}(0) \iff \partial_t : \text{gr}_1^V M_f \to \text{gr}_0^V M_f$ is surjective.

3. $\text{gr}_1^V M_f = \ker(t : \text{gr}_0^V M_f \to \text{gr}_1^V M_f) \oplus \text{im}(\partial_t : \text{gr}_1^V M_f \to \text{gr}_0^V M_f) \iff \mathcal{M} \simeq \mathcal{M}' \oplus \mathcal{M}''$ where $\text{supp} \mathcal{M}' \subset f^{-1}(0)$ and $\mathcal{M}''$ does not have nonzero subobjects or quotient objects whose support is contained in $f^{-1}(0)$.

**Proof.** We have

$$0 \to H^{-1} i^* M_f \to \text{gr}_1^V M_f \xrightarrow{\partial_t} \text{gr}_0^V M_f \to H^0 0^* M_f \to 0$$

$$0 \to H^0 0^* M_f \to \text{gr}_0^V M_f \xrightarrow{t} \text{gr}_1^V M_f \to H^1 0^* M_f \to 0$$

where $i : f^{-1}(0) \hookrightarrow X$. Thus (1), (2) are clear. For (3), first suppose the latter condition. Then $\text{gr}_1^V M_f = 0, \text{gr}_0^V M_f \simeq \ker(t : \mathcal{M}'_f \to \mathcal{M}'_f) = \ker(t : \text{gr}_0^V M_f \to \text{gr}_1^V M_f)$ as supp $\mathcal{M}' \subset f^{-1}(0)$.

(Recall that the $V$-filtration on $\mathcal{M}'_f$ is given by

$$V_0 \mathcal{M}'_f = \ker(t^{[\alpha + 1]} : \mathcal{M}'_f \to \mathcal{M}'_f) = \bigoplus_{0 \leq i \leq |\alpha|} (\mathcal{M}'_f)_i$$

where $(\mathcal{M}'_f)_0 = \ker(t : \mathcal{M}'_f \to \mathcal{M}'_f)$ and $(\mathcal{M}'_f)_i = (\mathcal{M}'_f)_0 \partial_t^i$. Also $t : \text{gr}_0^V \mathcal{M}''_f \to \text{gr}_1^V \mathcal{M}_f$ is injective and $\partial_t : \text{gr}_1^V \mathcal{M}''_f \to \text{gr}_0^V \mathcal{M}_f$ is surjective. Thus

$$\ker(t : \text{gr}_0^V \mathcal{M}_f \to \text{gr}_1^V \mathcal{M}_f) \oplus \text{im}(\partial_t : \text{gr}_1^V \mathcal{M}_f \to \text{gr}_0^V \mathcal{M}_f)$$

$$= \ker(t : \text{gr}_0^V \mathcal{M}_f \to \text{gr}_1^V \mathcal{M}_f) \oplus \text{im}(\partial_t : \text{gr}_1^V \mathcal{M}_f \to \text{gr}_0^V \mathcal{M}_f)$$

$$\oplus \ker(t : \text{gr}_0^V \mathcal{M}''_f \to \text{gr}_1^V \mathcal{M}_f) \oplus \text{im}(\partial_t : \text{gr}_1^V \mathcal{M}''_f \to \text{gr}_0^V \mathcal{M}_f)$$

$$= \text{gr}_0^V \mathcal{M}_f \oplus \text{gr}_0^V \mathcal{M}''_f = \text{gr}_0^V \mathcal{M}_f$$

as desired. Proving the converse is tricky; one may refer to [Sai88, Lemme 5.1.4].

Therefore...
Lemma 4.4. Suppose the difference between the definition of quasi-unipotency and the condition of the lemma already know that
\[ \partial V \, V = \partial f : V \to V, \]
along \( f \).

\[ \sum_{i=0}^{\infty} (F_{p-i}V_{<0}M) \partial^i \]

\[ \text{for every locally defined holomorphic function } f. \]

4. Compatibility of Nearby and Vanishing Cycles and the Filtration

We define the following.

Definition 4.1. We say that \( M = (M, F_\bullet M, K) \) is quasi-unipotent along \( f = 0 \) if all eigenvalues of the monodromy operator on \( \psi_f K \) are roots of unity, and if the \( V \)-filtration \( V_\bullet M_f \) satisfies the following two additional conditions.

1. \( t : F_p V_\alpha M_f \to F_p V_{\alpha-1} M_f \) is surjective for \( \alpha < 0 \).
2. \( \partial : F_p \, gr_\alpha^V M_f \to F_{p+1} \, gr_{\alpha+1}^V M_f \) is surjective for \( \alpha > 0 \).

Also, we say that \( M \) is regular along \( f = 0 \) if \( F_\bullet M_f \) is a good filtration for every \(-1 \leq \alpha \leq 0\).

Thus if \( M \) is regular along \( f = 0 \) then \( \psi_f M, \psi_{f,1} M, \phi_{f,1} M \) are well-defined as filtered regular holonomic \( D \)-modules with \( \mathbb{Q} \)-structure. But what does it mean that \( M \) is quasi-unipotent along \( f = 0 \)? It will be clear as we observe the following lemmas.

Lemma 4.2. Suppose \( M \) has strict support \( Z \subset X \) and \( f \) is nonconstant on \( Z \). Let \( j : X \times (\mathbb{C} - \{0\}) \to X \times \mathbb{C} \) be the inclusion map and \( F_p V_{<0} M_f = V_{<0} M_f \cap F_p M_f \). Then \( F_p V_{<0} M_f = V_{<0} M_f \cap j_* j^* F_p M_f \) if and only if \( t : F_p V_\alpha M_f \to F_p V_{\alpha-1} M_f \) is surjective for every \( \alpha < 0 \).

Lemma 4.3. Suppose \( M \) has strict support \( Z \subset X \) and \( f \) is nonconstant on \( Z \). Also suppose \( \partial : gr_{\alpha-1}^V M \to gr_\alpha^V M \) is surjective. (It indeed follows from strict support condition.) Then \( F_p M = \sum_{i=0}^{\infty} (F_{p-i} V_{<0} M) \partial^i \) if and only if \( \partial : F_p V_\alpha^V M \to F_{p+1} V_{\alpha+1} M \) is surjective for \( \alpha \geq -1 \).

Recall that \( M_f = \sum_{i=0}^{\infty} (V_0 M_f) \partial^i \), i.e. \( M_f \) is generated by \( V_0 M_f \) as a \( D \)-module. As we already know that \( \partial : gr_{\alpha-1}^V M \to gr_\alpha^V M \) is surjective, we have \( M = \sum_{i=0}^{\infty} (V_{<0} M_f) \partial^i \). Remark that \( V_{<0} M_f \) depends on the restriction of \( M_f \) to \( X \setminus X_0 \). Thus if we combine two lemmas above, we see that \( F_p M_f \) is completely determined by the data on \( X \setminus X_0 \), i.e.

\[ F_p V_{<0} M_f = V_{<0} M_f \cap j_* j^* F_p M_f = V_{<0} M_f \cap j_* j^* \sum_{i=0}^{\infty} (F_{p-i} V_{<0} M_f) \partial^i = \sum_{i=0}^{\infty} (V_{<0} M_f \cap j_* j^* F_{p-i} M_f) \partial^i. \]

Note the difference between the definition of quasi-unipotency and the condition of the lemma above; after we introduce Hodge modules it would become clear.

Now what if \( Z \subset f^{-1}(0) \)?

Lemma 4.4. Suppose supp \( M \subset f^{-1}(0) \) and set \( M_0 = \ker(t : M \to M) \) and \( F_p M_0 = F_p M \cap M_0 \). Then \( M \cong M_0 \oplus \partial \) and \( F_p M \cong \bigoplus_{i=0}^{\infty} F_{p-i} M_0 \otimes \partial^i \) if and only if \( \partial : F_p V_\alpha^V M \to F_{p+1} V_{\alpha+1}^V M \) is surjective for \( \alpha > -1 \).

Lemma 4.5. Suppose supp \( M \subset f^{-1}(0) \). Then \( M = (M, F_\bullet M, K) \) is quasi-unipotent and regular along \( f = 0 \) if and only if \( (F_p M) f \subset F_{p-1} M \) for all \( p \in \mathbb{Z} \).
Thus the definition above has a meaning even when \( \text{supp} M \subset f^{-1}(0) \). Indeed, we use this definition to decompose filtered regular holonomic \( \mathcal{D} \)-modules with \( \mathbb{Q} \)-structure as follows.

**Proposition 4.6.** Let \( M \) be a filtered regular holonomic \( \mathcal{D} \)-modules with \( \mathbb{Q} \)-structure which is quasi-unipotent and regular along \( f = 0 \) for every locally defined holomorphic function \( f \). Then

\[
M = \bigoplus_{Z \subset X} M_Z
\]

where \( M_Z \) is a filtered regular holonomic \( \mathcal{D} \)-module with \( \mathbb{Q} \)-structure which has strict support \( Z \), if and only if

\[
\phi_{f,1}M = \ker(\text{var} : \phi_{f,1}M \rightarrow \psi_{f,1}M(-1)) \oplus \text{im}(\text{can} : \phi_{f,1}M \rightarrow \psi_{f,1}M)
\]

for every locally defined holomorphic function \( f \). Here the filtration of \( \text{im}(\text{can} : \phi_{f,1}M \rightarrow \psi_{f,1}M) \) is induced from that of \( \phi_{f,1}M \).

5. **Definition of Pure Hodge Modules**

Here comes our main object.

**Definition 5.1.** Consider the category of filtered regular holonomic \( \mathcal{D} \)-modules with \( \mathbb{Q} \)-structure that are quasi-unipotent and regular along \( f = 0 \) for any locally defined holomorphic \( f \), and that are direct sums of objects which have strict support. Then we define the category of (pure) Hodge modules of weight \( w \), denoted \( \text{HM}(X, w) \), to be the largest full subcategory of the above such that

1. if \( M \) has support \( \{x\} \subset X \), then \( M = i_x^!(H^\bullet_{\mathbb{C}}F^\bullet_{\mathbb{C}}H^\bullet_{\mathbb{Q}}) \) where \( i_x : \{x\} \hookrightarrow X \) is the inclusion, and \( (H^\bullet_{\mathbb{C}}, F^\bullet_{\mathbb{C}}H^\bullet_{\mathbb{Q}}) \) is a rational Hodge structure of weight \( w \),

2. for all Zariski open \( U \subset X \), \( f : U \rightarrow \mathbb{C} \) a non-constant holomorphic function, and \( M \) having support not contained in \( f^{-1}(0) \), if \( M \in \text{HM}(X, w) \), then for all \( k \in \mathbb{Z} \) the graded modules \( \text{gr}^W_k \psi_f M \) and \( \text{gr}^W_k \phi_{f,1}M \) are Hodge modules of weight \( k \) with support on \( f^{-1}(0) \) where \( W \) is a monodromy weight filtration.

Then we have the following properties.

**Proposition 5.2.**

1. \( \text{HM}(X, w) \) is a direct sum \( \bigoplus_{Z \subset X} \text{HM}_Z(X, w) \) where \( \text{HM}_Z(X, w) \) is the category of Hodge modules of weight \( w \) on \( X \) with strict support on \( Z \).

2. Every morphism between two Hodge modules is strictly compatible with the filtration, i.e. the map from coimage to image is an isomorphism of filtered objects.

3. \( \text{HM}(X, w) \) is an abelian category.

4. Every morphism from one Hodge module to the other of strictly smaller weight is trivial.

Now consider \( \partial_t : F_p \text{gr}^V_{-1}M_f \rightarrow F_{p+1} \text{gr}^V_0 M_f \). By strict support condition we already know that \( \partial_t : \text{gr}^V_{-1}M_f \rightarrow \text{gr}^V_0 M_f \) is surjective. As \( \partial_t \) is strictly compatible with the filtration, it implies \( \partial_t : F_p \text{gr}^V_{-1}M_f \rightarrow F_{p+1} \text{gr}^V_0 M_f \) is also surjective. Compare this result with the definition of quasi-unipotency along \( f = 0 \).
6. Polarization

We want to define polarized Hodge modules. To that end it is natural to consider $K \otimes K \to \mathbb{Q}(-w)[2n]$, or equivalently, $K(w) \to \mathbb{D}K$. Also we can define $\mathcal{D}M$, so that we can consider $\mathcal{M} \to \mathbb{D}\mathcal{M}$. But the problem is that it does not behave well with the filtration: we cannot just take the Verdier dual of the filtration. Fortunately, we can define a good notion of dual $\mathbb{D}M \in H^\bullet_{\mathbb{Q}}(X, -w)$ such that $rat\mathbb{D}M = \mathbb{D}K$ is the usual Verdier dual of $K$. (It is related to the Cohen-Macaulayness of $gr^B\mathcal{F} \mathcal{M}$ over $gr^B\mathcal{D}X$, proved in [Sai88, Lemme 5.1.13].)

Now the definition:

**Definition 6.1.** A polarization on a Hodge module $M \in H^\bullet_{\mathbb{Q}}(X, w)$ is a morphism $K(w) \to \mathbb{D}K$ such that

1. it is nondegenerate and compatible with the filtration, thus it extends to $M(w) \simeq \mathbb{D}M$,
2. if $M$ has strict support $Z \subset X$ and $U$ is Zariski open dense in $X$ such that $Z \cap U \neq \emptyset$, if a holomorphic $f : U \to \mathbb{C}$ is not constant on $Z \cap U$, then the induced morphism $pr^\bullet f K(w) \to \mathbb{D}(pr^\bullet f K)$ gives a polarization on primitive parts for $N$,
3. if $M$ is supported on a point, then $K(w) \to \mathbb{D}K$ is induced by a polarization of Hodge structures.

If $M$ admits one polarization then we say that $M$ is polarizable. Also we denote by $H^\bullet_{\mathbb{P}}(X, w)$, $H^\bullet_{\mathbb{P}}(X, w)$ the full subcategory of polarizable Hodge modules in $H^\bullet_{\mathbb{Q}}(X, w)$, $H^\bullet_{\mathbb{Z}}(X, w)$, respectively.

7. On Singular Spaces

If $X$ is not smooth, then we can use Kashiwara’s theorem. Thus if $X$ can be embedded to smooth $Y$, then we define $H^\bullet_{\mathbb{Q}}(X, w)$, $H^\bullet_{\mathbb{P}}(X, w)$ to be full subcategories of $H^\bullet_{\mathbb{Q}}(Y, w)$, $H^\bullet_{\mathbb{P}}(Y, w)$ whose objects are (polarizable) Hodge modules supported on $X$. Note that Kashiwara’s theorem does not work for all the filtered regular holonomic $\mathcal{D}$-modules with $\mathbb{Q}$-structure as a coherent sheaf with support in $X$ is not the same thing as a coherent sheaf on $X$, thus the notion of good filtration differs. But it is still true if we consider modules which are quasi-unipotent and regular along $f = 0$ for any locally defined holomorphic $f$. Also the definition of $H^\bullet_{\mathbb{Q}}(X, w)$ is independent of the choice of embedding, thus it is well-defined. Note that if $X$ is not globally embedded and we can locally embed $X$ and glue categories together to get $H^\bullet_{\mathbb{Q}}(X, w)$.

References


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, U.S.A.

E-mail address: sylvaner@math.mit.edu