AN OVERVIEW OF THE DECOMPOSITION THEOREM

DONGKWAN KIM

1. Statement and Proof of the Main Theorem

1.1. The Statement of the Main Theorem. We first state the main theorem.

**Theorem 1** (Decomposition theorem). Let \( f : X \to Y \) be a proper morphism of algebraic varieties over \( \mathbb{C} \). Then we have

\[
 f_* IC_X = \sum_{i \in \mathbb{Z}} p^i \mathcal{H}^i(f_* IC_X)[-i]
\]

where each \( p^i \mathcal{H}^i(f_* IC_X) \) is a semisimple perverse sheaf.

There are currently three known approaches to prove this theorem.

1. The original proof by Beilinson, Bernstein, Deligne, and Gabber [BBD]
2. Using mixed Hodge modules by Saito [S]
3. Using classical Hodge theory by de Cataldo and Migliorini [dCM2]

Here we briefly sketch the original proof of [BBD]. The main reference of this talk would be [dCM1], the expository article for this subject written by de Cataldo and Migliorini.

The main idea of [BBD] is that first we prove this theorem over a finite field and use “spread out” principle. One of the important property of this method is that we have an action of the Galois group which acts on varieties and sheaves, which is topologically generated by the Frobenius morphism. Here we introduce the notion of weights; it is basically the absolute value of its eigenvalues. If the weights of some complex is in sufficiently good shape, we call it mixed/pure, and we will see that it indeed gives a rigid structure on such complexes and morphisms between them.

1.2. Some notations. First we work over a finite field. Let \( \mathbb{F}_q \) be a finite field, \( \mathbb{F} = \overline{\mathbb{F}_q} \) be its algebraic closure, \( X_0 \) be an algebraic variety over \( \mathbb{F}_q \), and \( X = X_0 \times_{\mathbb{F}_q} \mathbb{F} \). Also we let \( F \) be the geometric Frobenius with respect to \( \mathbb{F}_q \). We fix a prime \( \ell \) prime to \( q \), and consider the bounded constructible derived category \( \mathcal{D}_c(X_0, \overline{\mathbb{Q}_\ell}) \). We usually put subscript \( 0 \) to denote analogous objects over \( \mathbb{F}_q \) corresponding to some objects over \( \mathbb{F} \).

**Definition 2** ([BBD 5.1.5]). For a \( \overline{\mathbb{Q}_\ell} \) sheaf \( \mathcal{F}_0 \) on \( X_0 \), we call it pointwise/punctually pure of weight \( w \in \mathbb{Z} \) when for any \( n \geq 1 \) and for any \( x \in X_0(\mathbb{F}_q^n) \), all the eigenvalues and their complex conjugates of \( (F^n)^* \) on \( \mathcal{F}_x \) have absolute value \( q^{nw/2} \). We call \( \mathcal{F}_0 \) mixed when it admits a finite filtration of pointwise pure complexes. The weights of these nonzero successive quotients are called pointwise weights of \( \mathcal{F}_0 \). We denote by \( \mathcal{D}_c^m(X_0, \overline{\mathbb{Q}_\ell}) \) the subcategory of \( \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_\ell}) \) consisting of complexes whose cohomology sheaves are mixed, which we call mixed complexes.

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One can prove that $D^b_m(X_0, \mathbb{Q}_\ell)$ is stable under $f_*, f^!, f^*, f^! \otimes, \mathcal{H}om$, and Verdier duality. Also it is stable under the usual truncation functor. Since the perverse t-structure can be defined using ”glueing” procedure, we also see that it is stable under perverse truncations. [BBD, 5.1.6] Thus $D^b_m(X_0, \mathbb{Q}_\ell)$ is also naturally a triangulated subcategory in both senses. Also, the category of mixed perverse sheaves, often denoted by $\mathcal{P}_m(X_0, \mathbb{Q}_\ell)$, is abelian. One can also show that $\mathcal{P}_m(X_0, \mathbb{Q}_\ell)$ is closed under taking subquotients. [BBD, 5.1.7]

**Definition 3 ([BBD, 5.1.8]).** For $K \in D^b_m(X_0, \mathbb{Q}_\ell)$, we say it is of weight $\leq w$ if the pointwise weights of $H^i K$ is $\leq w + i$ for all $i \in \mathbb{Z}$. Also we say it is of weight $\geq w$ if its Verdier dual $\mathbb{D}K$ is of weight $-w$. If $K$ is of weight $\leq w$ and $\geq w$, then we call it pure of weight $w$.

**Example 4.** if $X_0$ is smooth and irreducible of dimension $d$ and $L[d]$ is a lisse sheaf on $X_0$ concentrated on degree $-d$, then it is pure of weight $w$ if $H^i L[d] = \mathcal{L}^\vee(d)$ is of pointwise weight $\leq -w + d$, hence $\mathcal{L}^\vee$ is of pointwise weight $\leq w + d$. (Note the Tate twist.) But it means that $\mathcal{L}$ is of pointwise weight exactly $w - d$.

1.3. **Weight and Six Functors.**

**Proposition 5 ([BBD, 5.1.14]).** Let $F_0$ be of weight $\leq w$, $F'_0$ be of weight $\leq w'$, and $G''_0$ be of weight $\geq w''$. Then

1. $f_! F_0, f^* F_0$ is of weight $\leq w$.
2. $f^! G_0, f_* G_0$ is of weight $\geq w''$.
3. $F_0 \otimes F'_0$ is of weight $\leq w + w'$.
4. $\mathcal{H}om(F_0, G_0)$ is of weight $\geq -w + w''$.
5. $\mathbb{D} F_0$ is of weight $\geq -w$ and $\mathbb{D} G_0$ is of weight $\leq -w''$.

**Proof.** $f^*$ one is obvious. Since it is clear that exterior tensor product sums up weights with the same direction, (3) is also clear. (5) is by definition. Then $f_!$ one also follows. Since $\mathcal{H}om(F_0, G_0) \simeq \mathbb{D} F_0 \otimes_f G_0$, (4) follows. Now $f_*, f_!$ ones follow from adjunction. Or we can use [D, 3.3.1, 6.2.3] by a dévissage argument to the cohomology with compact support and use of Leray spectral sequence. □

**Remark.** Note that this implies Riemann hypothesis part of the Weil conjecture. Suppose $X$ is a smooth projective variety of dimension $d$ over $\mathbb{F}_q$ and consider the following zeta function:

$$\zeta(X, s) = \exp \left( \sum_{n=1}^{\infty} \frac{\# X(\mathbb{F}_{q^n})}{n} q^{-ns} \right)$$

By Lefschets fixed point formula, we have

$$\# X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^i \text{tr}((F^n)^*, H^i_c(X, \mathbb{Q}_\ell))$$
If we denote $\lambda_{i,1}, \cdots, \lambda_{i,j}$ the eigenvalues of $F^*$ on $H^q_c(X, \mathbb{Q}_\ell)$, then we have

$$
\zeta(X, s) = \exp \left( \sum_{i=0}^{2n} \left( \sum_{n=1}^{\infty} (-1)^i \frac{\lambda_{i,1}^n + \cdots + \lambda_{i,j}^n}{n} q^{-ns} \right) \right)
$$

$$
= \prod_{i=0}^{2n} \exp \left( (-1)^i \log(1 - \lambda_{i,1}^{-s}) \cdots \log(1 - \lambda_{i,j}^{-s}) \right)
$$

$$
= \frac{P_1(q^{-s}) \cdots P_{2n-1}(q^{-s})}{P_0(q^{-s}) P_2(q^{-s}) \cdots P_{2n}(q^{-s})}
$$

where

$$
P_i(q^{-s}) = (1 - \lambda_{i,1}^{-s}) \cdots \log(1 - \lambda_{i,j}^{-s}).
$$

Since the action of $F$ is of finite order, $\lambda_{i,j}$ are algebraic integers, thus $P_i(q^{-s})$ is an integral polynomial. Then by the property of $f_*$ and $f_1$, $\lambda_{i,j}$ (and its complex conjugates) has eigenvalues whose absolute values are $q^{i/2}$ since $\mathbb{C}_X$ is pure of weight 0.

1.4. Semisimplicity.

**Proposition 6** ([BBD 5.1.15]). Let $K_0, L_0 \in D^b_m(X_0, \mathbb{Q}_\ell)$ and suppose $K_0$ is of weight $\leq w$ and $L_0$ is of weight $> w$. Then for $a : X_0 \to \text{Spec} \mathbb{F}_q$, we have

1. $a_* \text{Hom}(K_0, L_0)$ is of weight $> 0$.
2. $\text{Hom}^i(K_0, L_0) = 0$ for $i > 0$.

This time $L_0$ is of weight $\geq w$, then

3. $\text{Hom}^i(K, L)^F = 0$ for $i > 0$. In particular, the morphism $\text{Hom}^i(K_0, L_0) \to \text{Hom}^i(K, L)$ is trivial for $i > 0$.

**Proof.** The first assertion comes from the fact that $\text{Hom}(K_0, L_0)$ is of weight $> 0$ (resp. $\geq 0$). Also we note that there exists an exact sequence

$$
0 \to \text{Hom}^{i-1}(K, L)_F \to \text{Hom}^i(K_0, L_0) \to \text{Hom}^i(K, L)^F \to 0
$$

which comes from the spectral sequence

$$
E_2^{p,q} = H^p(\text{Spec} \mathbb{F}_q, \mathcal{H}^q M_0) = H^p(\text{Gal}(\mathbb{F}/\mathbb{F}_q), \mathcal{H}^q M) \Rightarrow H^{p+q}(\text{Spec} \mathbb{F}_q, M_0).
$$

Now (2) follows from this exact sequence, since for $i > 0$ there is no invariance on $\text{Hom}^i(K, L)$ and no coinvariance on $\text{Hom}^{i-1}(K, L)$. (3) also directly follows from this sequence. \qed

We cannot emphasize enough this proposition; if we put $i = 1$, then the vanishing of $\text{Hom}^1(K, L)$ means that there is no nontrivial extension. This combines with the concept of purity to give a condition of semi-simplicity on the decomposition theorem as follows.

**Theorem 7** ([BBD 5.3.8]). If $\mathcal{F}_0$ is a pure perverse sheaf on $X_0$, then $\mathcal{F}$ on $X$ is semisimple.

**Proof.** Let $\mathcal{F}' \subset \mathcal{F}$ be the maximal semisimple subsheaf. Then it is $F$-stable, thus it corresponds to some $\mathcal{F}_0' \subset \mathcal{F}_0$. We have an exact sequence

$$
0 \to \mathcal{F}_0' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}_0' \to 0.
$$

But $\mathcal{F}_0'$ and $\mathcal{F}_0/\mathcal{F}_0'$ have the same weight, from which we know that

$$
\text{Hom}^1(\mathcal{F}_0', \mathcal{F}_0/\mathcal{F}_0') \to \text{Hom}^1(\mathcal{F}', \mathcal{F}/\mathcal{F}')
$$

is a zero map. Thus $\mathcal{F}/\mathcal{F}' = 0$ by maximality and we obtain the result. \qed
Note that this is false on $X_0$ itself since $F$ can have "nontrivial Jordan blocks." In a similar spirit, one can prove the following theorem of which we will omit the proof.

**Theorem 8 (BBD 5.4.6).** If $F_0 \in D^b_m(X_0, \overline{\mathbb{Q}_\ell})$ is pure, then $F$ is the direct sum of simple perverse sheaves of (possibly) shifted degree.

Now it remains to show that $IC$ sheaves are pure.

### 1.5. Purity of IC Complexes: Gabber’s Theorem.

**Proposition 9 (BBD 5.3.1).** For a mixed perverse sheaf of weight $\leq w$ (resp. $\geq w$), its subquotients are also mixed of weight $\leq w$ (resp. $\geq w$).

**Theorem 10 (BBD 5.3.2), Gabber purity.** Let $j : U_0 \hookrightarrow X_0$ be an (affine) open embedding. Then if $F_0$ is a mixed perverse sheaf of weight $\leq w$ (resp. $\geq w$) then $j_* F_0$ is also mixed of weight $\leq w$ (resp. $\geq w$). Therefore, especially, if $F_0$ is pure, then $j_* F_0$ is pure of the same weight.

If $j$ is affine then $j_* F_0$ is a quotient of $j_! F_0$ which is also perverse and mixed, thus the theorem follows from the proposition above. In general it is trickier.

**Corollary 11 (BBD 5.3.4).** Mixed simple perverse sheaves are pure.

Every simple perverse sheaf is of the form $j_! (L_0 [d])$. If it is mixed, then $L_0$ is also mixed. But it means it is pointwise pure since it is irreducible. Thus the result follows.

We see now that we proved the decomposition theorem over $\mathbb{F}$; indeed, we have

1. $IC$ sheaves are pure by Gabber’s theorem.
2. The pushforward of pure complexes by a proper map is also pure.
3. A pure complex is, after base change to $\mathbb{F}$, a direct sum of simple perverse sheaves of shifted degree.

### 1.6. From $\mathbb{F}$ to $\mathbb{C}$: Spread Out.

To prove the result over $\mathbb{C}$, we use "spread out" principle, i.e. if it is true for infinitely many primes, i.e. on a dense set of $\text{Spec} \mathbb{Z}$, then it is also true over $\text{Spec} \mathbb{C}$. This principle is also used to prove the algebraic proof of the degeneration of Hodge-to-de Rham spectral sequence by Deligne and Illusie. Here we simply state some notations and the decomposition theorem over $\mathbb{C}$ without proof.

**Definition 12 (BBD 6.2.4).** Let $X$ be an algebraic variety over $\mathbb{C}$. For a simple perverse $\mathbb{C}$-sheaf over $X(\mathbb{C})$, we say that it is of geometric origin if it belongs to the smallest set which satisfies the following.

1. It contains $\mathbb{C}_{pt}$.
2. It is stable under taking a simple constituent of $p^* \mathcal{H}^i$ of $f_*, f_! , f^* , f^! , \otimes, \mathcal{H}om$.

**Definition 13 (BBD 6.2.4).** For $F \in D^b(X(\mathbb{C}), \mathbb{C})$, we say it is semisimple of geometric origin if it is a direct sum of simple perverse sheaves of geometric origin with possible degree shift.

**Theorem 14 (BBD 6.2.5).** If $f : X \to Y$ is proper, then $f_*$ sends semisimple complexes of geometric origin to ones of the same kind.
2. Semi-small and Small Maps

2.1. Definition of Semi-small Maps. We recall the notion of semi-small maps. Note that we usually assume that the source of the semi-small map is smooth.

**Definition 15.** For a proper surjective morphism \( f : X \to Y \) where \( X \) is irreducible and smooth, we say it is **semi-small** if the following equivalent conditions hold. (\( n = \dim X \))

1. \( f_* \mathbb{C}_X[d] \) is perverse.
2. \( \dim X \times_Y X \leq n \) (or equivalently = \( n \))
3. \( \dim \{ y \in Y \mid \dim f^{-1}(y) \geq i \} \leq \dim X - 2i \) for all \( i \geq 0 \)

In particular, if it further satisfies

\( (3') \) \( \dim \{ y \in Y \mid \dim f^{-1}(y) \geq i \} < \dim X - 2i \) for all \( i > 0 \),

then we say it is **small**.

Note that it implies that \( f \) is generically finite, thus \( \dim X = \dim Y \).

**Definition 16.** Suppose \( f : X \to Y \) is semi-small and \( Y = \sqcup Y_\alpha \) is an stratification where \( f|_{f^{-1}Y_\alpha} \) is a (topological) fiber bundle. (Such a stratification always exists.) Then we define

\[ L_\alpha := \mathcal{H}^{\dim Y - \dim Y_\alpha}(f_* \mathbb{C}_X|_{Y_\alpha}). \]

Note that it is nonzero if and only if \( \dim Y - \dim Y_\alpha = 2 \dim f^{-1}(y) \) for some/any \( y \in Y_\alpha \). If so, we say that \( Y_\alpha \) is **relevant**.

Note that we have a decomposition

\[ L_\alpha = \bigoplus_{\phi_\alpha} L_{\phi_\alpha}^{\oplus m_{\phi_\alpha}} \]

where each of \( L_{\phi_\alpha} \) corresponds to the monodromy representation \( \phi_\alpha \) of \( \pi_1(Y_\alpha) \). Since \( (L_\alpha)_y = \mathcal{H}^{\dim Y - \dim Y_\alpha}(f_* \mathbb{C}_X|_y) \) is the top dimensional cohomology of the fiber of \( y \), i.e. \( H^{\dim Y - \dim Y_\alpha}(f^{-1}(y), \mathbb{C}_{f^{-1}(y)}) \), it has a basis which corresponds to the top dimensional irreducible components of \( f^{-1}(y) \). Then the monodromy on \( L_\alpha \) comes from the permutation action of \( \pi_1(Y_\alpha) \) on these components. In other words, we have

\[ H^{\dim Y - \dim Y_\alpha}(f^{-1}(y), \mathbb{C}_{f^{-1}(y)}) \simeq \bigoplus \mathbb{C}^{m_{\phi_\alpha}} \otimes \phi_\alpha \]

where \( m_{\phi_\alpha} \) is the multiplicity of \( \phi_\alpha \) on \( H^{\dim Y - \dim Y_\alpha}(f^{-1}(y), \mathbb{C}_{f^{-1}(y)}) \).

2.2. Decomposition Theorem for Semi-small Maps.

**Theorem 17.** We have

\[ f_* \mathbb{C}_X[d] = \bigoplus_{Y_\alpha} IC(\overline{Y_\alpha}, L_\alpha) = \bigoplus_{(Y_\alpha, \phi_\alpha)} IC(\overline{Y_\alpha}, L_{\phi_\alpha})^{\oplus m_{\phi_\alpha}} \]

where the direct sum is over all relevant strata and \( m_{\phi_\alpha} \) is as above. In particular, if \( f \) is small, then

\[ f_* \mathbb{C}_X[d] = IC(Y, \mathcal{L}). \]

where \( \mathcal{L} \) is with respect to the open dense stratum of \( Y \).
2.3. Some Examples.

Definition 18. For a normal variety \( X \), we say it has \textit{symplectic singularities} if there exists a closed nondegenerate 2-form \( \omega \) on the smooth locus of \( X \) such that for some (any) resolution of singularity \( \tilde{X} \to X \) the pull-back of \( \omega \) extends to a symplectic form on the whole of \( \tilde{X} \). In this case we call it a \textit{symplectic resolution} of \( X \).

Theorem 19 ([K1, 1.2], [K2, 2.11]). A symplectic resolution is semismall.

Corollary 20. The Springer resolution \( \tilde{N} \to N \) is semismall.

This is because \( \tilde{N} \cong T^*B \) has a natural symplectic structure. Also one can use another argument; indeed, it is not hard to prove that \( \tilde{N} \times_N \tilde{N} \) is a union of conormal bundles over each nilpotent orbit in \( N \), and thus it is enough to show that the number of such orbits are finite, e.g. [L]. Also this argument gives that every nilpotent orbit is relevant.

Likewise, we have the following property.

Proposition 21. The Grothendieck-Springer resolution \( \tilde{g} \to g \) is small.

3. Examples


Example 22. Suppose \( X \subseteq \mathbb{P}^n \) is an irreducible projective variety of dimension \( d \) and let \( Y \subseteq \mathbb{A}^{n+1} \) be its cone, and \( \pi : \tilde{Y} \to Y \) be the blowup at the origin. Then the decomposition theorem says \( \pi_*\mathbb{C}_Y[d+1] \) is semisimple, and also since \( \pi \) is an isomorphism away from the origin, we have

\[
\pi_*\mathbb{C}_Y[d+1] = IC_Y \oplus \mathcal{F}_0
\]

where \( \mathcal{F}_0 \) is supported at the origin. If we take the stalk at the origin on both sides, we get

\[
H^*(X, \mathbb{C}_X)[d+1] = (IC_Y)_0 \oplus \mathcal{F}_0
\]

Furthermore, note that \( \pi_*\mathbb{C}_Y[d] \) and \( IC_Y \) are self-dual, thus we have

\[
\pi_*\mathbb{C}_Y[d+1] = IC_Y \oplus \mathcal{F}_0^\vee.
\]

In other words, \( \mathcal{F}_0 \) is also self-dual. Now we note that \( (IC_Y)_0 \) sits on the degree in \([-d-1, -1]\). Thus we see that

\[
\mathcal{F}_0 \cong H^{2d}(X, \mathbb{C}_X)[-d+1] \oplus \cdots \oplus H^{d+1}(X, \mathbb{C}_X) \oplus \cdots \oplus H^{2d}(X, \mathbb{C}_X)[d-1]
\]

and rest of terms in \( H^*(X, \mathbb{C}_X)[d+1] \) is contained in \( (IC_Y)_0 \).

If we know \( IC_Y \) on 0, then we can calculate the intersection cohomology of \( Y \) using contraction principle. For example, if \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) as in Simon’s talk. Then we have

\[
\mathcal{F}_0 = \mathbb{C}[1] \oplus \mathbb{C}[-1] \text{ and } IC_Y = \mathbb{C}_Y \oplus \mathbb{C}_0[1].
\]

Thus we have that

\[
\dim IH^i(Y, \mathbb{C}) = 1 \text{ if } i = 0, 2, \quad 0 \text{ otherwise.}
\]

If we use another blowup of \( Y \) along the plane \( \mathbb{P}^1 \times \mathbb{A}^1 \), then the fiber at 0 is just \( \mathbb{P}^1 \) and this resolution is small. Thus we directly see that \( (IC_Y)_0 = \mathbb{C}[2] \oplus \mathbb{C}. \)
**Example 23.** Suppose a finite group $G$ acts freely on $X$. Then the categorical quotient $\pi : X \to X//G$ is finite thus small. Thus we have

$$\pi_* IC_X = IC(X//G, \mathcal{L})$$

for some $\mathcal{L}$ on $(X^{sm}//G) \subset (X//G)^{sm}$ since $X//G$ has only one relevant stratum. Furthermore, if we restrict it to the smooth locus $X^{sm}$, then the restriction of the LHS on $X^{sm}//G$ is just a vector bundle with the $G$-action where each fiber is the regular representation of $G$. Thus

$$\pi_* IC(X, \mathbb{C}_X) = \bigoplus_{\chi \in \hat{G}} IC(X//G, \mathcal{L}_\chi)^{m_\chi}$$

where $\mathcal{L}_\chi$ corresponds to the irreducible character $\chi$ of $G$.

**Example 24.** Suppose a finite group $G$ acts on a smooth variety $X$ of dimension $d$. Then for $\pi : X \to Y = X//G$ we see that $C_Y \to R^0\pi_* C_X$ splits which corresponds to $G$-invariants on $R^0\pi_* C_X$. Since $R^0\pi_* C_X[d] \simeq \pi_* C_X[d]$ and it is self-dual, we see that $C_Y[d]$ is also self-dual. Thus we can easily check that $C_Y[d] \simeq IC_Y$.

Here is a detailed example. Suppose we have an action of $\mathbb{Z}/3$ on $\mathbb{A}^2$ by

$$(x, y) \mapsto (\omega x, \omega^{-1} y)$$

where $\omega$ is the third root of unity. Then we have $\rho : \mathbb{A}^2 \to X = \mathbb{A}^2/\mathbb{Z}^3$ and

$$\rho_* C_{\mathbb{A}^2}[2] = \bigoplus_{\mathcal{L}_\alpha} IC(X, \mathcal{L}_\alpha)$$

where each $\mathcal{L}_\alpha$ corresponds to the representation of $\mathbb{Z}/3$ since $\rho$ is finite. (Note that on $\mathbb{A}^2 \setminus 0$ the action of $\mathbb{Z}/3$ is free.) Also direct calculation shows that the blowup $\pi : \tilde{X} \to X$ along 0 has fiber $\mathbb{P}^1 \setminus \mathbb{P}^1$ over 0. Thus we have

$$\pi_* C_{\tilde{X}}[2] \simeq IC_X \oplus \mathbb{C}^2 \text{ and } IC_X = \mathbb{C}_X[2].$$

Now it is clear that $IC(X, \mathcal{L}_\alpha)_0 = 0$ if $\alpha$ is not trivial.

**3.2. Springer Correspondence.** Originally the Springer correspondence tells that we can find every representation of the Weyl group $W$ of a reductive group $G$ by looking at the action of $W$ on the cohomology of each Springer fiber $B_x$ for some nilpotent element $x \in \mathfrak{g}$. Here we will find a geometric analogue of such correspondence.

Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. Then since this is semismall, we have

$$\pi_* C_{\tilde{\mathcal{N}}}[d] = \bigoplus_{\mathcal{O}_{\phi_O}} IC(\mathcal{O}, \mathcal{L}_{\phi_O})^{m_{\phi_O}}$$

as before. Now we take granted that this is also true if we impose $G$-equivariant conditions. That is, note that the stratification of $\mathcal{N}$ given by $G$-orbits are trivially $G$-equivariant and as $\mathbb{C}_{\mathcal{N}}[d]$ and $\pi$ are $G$-equivariant, each $\mathcal{L}_{\phi_O}$ should also be $G$-equivariant. Now for any $x \in \mathcal{O}$, we have a long exact sequence

$$\cdots \to \pi_1(G) \to \pi_1(\mathcal{O}) \to \pi_0(G_x) \to \pi_0(G) \to \pi_0(\mathcal{O}) \to 1$$

Since $\mathcal{L}_{\phi_O}$ are $G$-equivariant, if $t \in \pi_1(\mathcal{O})$ is on the image of $\pi_1(G)$, then the action of $t$ on each $\mathcal{L}_{\phi_O}$ is trivial. Thus we see that every $\phi_O$ factors through $\pi_0(G_x) = G_x/G_x^0$. Note that this action is basically the permutation action on the top dimensional cohomology of the
fiber of $O$, i.e. the irreducible components of the Springer fiber of $x$. (Note that Springer fibers are equidimensional.)

By Schur’s lemma, we take the endomorphism of both sides to get

$$\text{End}(\pi_\ast C_{\tilde{N}}[d]) = \bigoplus_{O, \phi} \text{End} \mathbb{C}^{m_{\phi O}}$$

In other words, $\text{End}(\pi_\ast C_{\tilde{N}}[d])$ is semisimple. Now we use the following lemma.

**Lemma 25** ([CH, 2.21(2) and 2.23]). We have an isomorphism

$$\text{End}(\pi_\ast C_{\tilde{N}}[d]) \cong \text{Hom}_{BM}^{2d}(\tilde{N} \times_{N} \tilde{N})$$

where the RHS is equipped with the usual convolution structure.

Meanwhile, note that we have the following diagram.

\[
\begin{array}{ccc}
\tilde{N} & \xrightarrow{i'} & \tilde{g} \\
\downarrow \pi & & \downarrow \pi' \\
N & \xrightarrow{i} & g
\end{array}
\]

Then we have $\pi_\ast C_{\tilde{N}}[d] = i'_{\ast} i'_{\ast} C_{\tilde{g}}[d] = i'_{\ast} IC(g, L)$ where $L$ is supported on $g^{rs}$ equipped with the natural action of the Weyl group $W$ since $\tilde{g}^{rs} \rightarrow g^{rs}$ is a principal $W$-bundle. Thus $\pi_\ast C_{\tilde{N}}[d]$ also has a natural $W$-action and this induces a homomorphism

$$\mathbb{C}[W] \rightarrow \text{End}(\pi_\ast C_{\tilde{N}}[d]).$$

Borho and MacPherson [BM] showed that this is indeed an isomorphism. Thus we have

$$\mathbb{C}[W] = \bigoplus_{O, \phi O} \text{End} \mathbb{C}^{m_{\phi O}}$$

which is the usual Springer correspondence.

### 3.3. Schubert Varieties and Kazhdan-Lusztig Polynomials. [H]

We will prove the following theorem of Kazhdan and Lusztig. Suppose $G$ is a reductive group over $\mathbb{F}_q$. For a flag variety $G/B$, we have the Bruhat decomposition $G/B = \sqcup_{w \in W} X(w)$. Then we have

**Theorem 26** ([KL, 4.2]). $H^i IC_{\tilde{X}(w)} = 0$ if $i$ is odd. If $i$ is even and $B' \in X(w)(\mathbb{F}_q)$, then all the eigenvalues of $F^n$ on $H^i_{BM} IC_{\tilde{X}(w)}$ are $q^{ni/2}$.

Here we use the Bott-Samelson resolution $\pi : \tilde{X}(w) \rightarrow X(w)$. For a reduced expression $w = s_1 \cdots s_l$, we define

$$\tilde{X}(w) = \{(g_1 B, \cdots, g_n B) \mid g_i^{-1} g_i \in B s_i B \text{ where } g_0 := id\}$$

and

$$\tilde{X}(w) \rightarrow X(w) : (g_1 B, \cdots, g_n B) \mapsto g_n B.$$ 

The variety $\tilde{X}(w)$ is an iterated $\mathbb{P}^1$-bundle, thus is smooth. Also, it is known that for any $y \in X(w)$, $\pi^{-1}(y)$ admits an affine paving. Thus by the decomposition theorem, first we see that $\pi_\ast IC_{\tilde{X}(w)}$ is semisimple. Also, it should contain $IC_{X(w)}$ as the restrictions of both on $X(w)$ coincide. Therefore, the first assertion follows since a variety paved by affine spaces
has vanishing odd cohomology. Also we see that all the eigenvalues of \((F^n)^*\) on \(H^{i}_B^{*} \pi_* \mathbb{Q}_{\tilde{X}(w)}\) are \(q^{ni/2}\).

From that we can prove that \(H^{i}_B^{*} \pi_* \mathbb{Q}_{\tilde{X}(w)}\) also has the same property. But note that this is not a consequence of the decomposition theorem; the theorem only works over \(\mathbb{F}\), not over \(\mathbb{F}_{q^n}\). Instead, one can use the distinguished triangle

\[
p^* \pi_* \mathbb{Q}_{\tilde{X}(w)} \to p^* \tau \geq j \pi_* \mathbb{Q}_{\tilde{X}(w)} \to p^* \tau \geq j + 1 \pi_* \mathbb{Q}_{\tilde{X}(w)}
\]

and the fact that this becomes the direct sum after we base change from \(\mathbb{F}_{q^n}\) to \(\mathbb{F}\). It is even more nontrivial to show \(IC_{\tilde{X}(w)}\) has the same property, since again the decomposition only works over \(\mathbb{F}\). One needs to prove that \(IC_{\tilde{X}(w)}\) is a subquotient of \(p^* \pi_* \mathbb{Q}_{\tilde{X}(w)}\), e.g. \([GH, 10.7]\).

**References**


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, U.S.A.

E-mail address: sylvaner@math.mit.edu