1 Proof of Kashiwara’s Theorem

**Theorem 1.1** (Kashiwara). Let \( i : X \to Y \) be a closed embedding. Then \( i_* \) is an equivalence between \( \mathcal{M}'(D_X) \) and \( \mathcal{M}'_X(D_Y) \), where \( \mathcal{M}'_X(D_Y) \) is the category of right \( D_Y \)-modules set-theoretically supported on \( X \), i.e. for any \( f \in \mathcal{J}_X \) and \( m \in M \), \( mf^N = 0 \) for some \( N \in \mathbb{N} \).

To prove this, we first show the necessary condition.

**Theorem 1.2.** The image of \( i_* : \mathcal{M}'(D_X) \to \mathcal{M}'(D_Y) \) is contained in \( \mathcal{M}'_X(D_Y) \).

**Proof.** Indeed, \( i_*(M) = M \otimes_{D_X} D_{X \to Y} \), and \( D_{X \to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} D_Y = D_Y / JD_Y \) for a closed immersion. So we only need to prove this for \( D_{X \to Y} \). But note that the \( D_Y \) action is defined on the right, that is we need to prove for any \( m \in D_{X \to Y} \), \( mJ^N = 0 \) for some \( N \in \mathbb{N} \). But it is still true by using commutator relation, i.e. for any \( \partial^N, \partial^N f^{N+1} = \partial^{N-1} f^{N+1} \partial + \partial^{N-1} (N+1) f' f^N = \cdots = f(\partial^N + \cdots) \), whence the assertion.

Now we define a functor \( i^0 : \mathcal{M}'(D_Y) \to \mathcal{M}'(D_X) \) as

\[
i^0M = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, M) = \{ m \in M \mid mJ = 0 \}\]

To give a right \( D_X \)-module structure, for any vector field \( v \in \text{Vec}(X) \) you need to define something like \( mv \) such that \( mvJ = 0 \). For this, we need the following lemma.

**Lemma 1.3.** For any \( v \in \text{Vec}(X) \), we can locally find \( \tilde{v} \in \text{Vec}(Y) \) such that \( \tilde{v}|_X = v \) and \( \tilde{v} \) preserves \( J \).

**Proof.** Since \( X \) is smooth, it is a local complete intersection [1, Theorem 8.17, Example 8.22.1]. Thus for any \( x \in X \), there exists \( U \subset X \) and an étale coordinate system \( x_1, \ldots, x_k, y_1, \ldots, y_l \) such that \( X \cap U = V(y_1, \ldots, y_l) \) and \( x_1, \ldots, x_k \) gives an étale coordinate system on \( X \). Now it is clear that any vector field on \( X \) can be lifted locally, and they indeed preserve \( J \).
Now we define \( m \cdot v = m \tilde{v} \). It is invariant under the choice of the lifting, since if \( \tilde{v}' \) is another lifting we have \( m(\tilde{v} - \tilde{v}') = 0 \) since \( \tilde{v} - \tilde{v}' \in \mathcal{J} Vec(Y) \). (The restriction on \( X \) is zero. Use the local chart around the point as before!) Also \( m v \mathcal{J} = 0 \) is zero since \( v \) preserves \( \mathcal{J} \); indeed, for any \( f \in \mathcal{J} \), \( m \mathcal{J} f = m[v, f] + mfv = 0 \) since \( [v, f] = v(f) \in \mathcal{J} \).

Why do we define this functor?

**Theorem 1.4.** (1) \( i^0 \) is the right adjoint to \( i_{*0} \). i.e. \( i_{*0} : \mathcal{M}^{r}(\mathcal{D}_{X}) \leftrightarrow \mathcal{M}^{r}(\mathcal{D}_{Y}) : i^0 \).

(2) If we restrict \( i^0 \) to \( \mathcal{M}^{r}_{X}(\mathcal{D}_{Y}) \), then they are mutually inverses.

**Proof.** \( \text{Hom}_{\mathcal{D}_{Y}}(i_{*0}M, N) = \text{Hom}_{\mathcal{D}_{X}}(M, i^{0}N)? \) \( f \in \text{Hom}_{\mathcal{D}_{Y}}(i_{*0}M, N) \), define \( \tilde{f} : M \to i_{*0}M \to N \) by \( M \to i_{*0}M : m \mapsto m \otimes 1 \in M \otimes_{\mathcal{D}_{X}} \mathcal{D}_{X} \to \mathcal{D}_{Y} \). Since for \( a \in \mathcal{J} \) and \( m \in M \), \( \tilde{f}(m) = a = f(m \otimes 1) = f([m, a]) = 0 \), \( f(M) \subset i^{0}N \). Also for any \( v \in \text{Vec}(X) \), \( \tilde{f}(mv) = f(m \otimes \tilde{v}) = f([m, \tilde{v}] = 0 \) where \( \tilde{v} \) is the lifting of \( v \) on \( Y \) (locally.) (Note the definition of \( \mathcal{D}_{X} \)-action on \( i^{0}N \!\) ! Also \( v \cdot (1 \otimes 1) = 0 + 1 \otimes v \) by the \( \mathcal{D}_{X} \)-module structure of \( \mathcal{D}_{X} \to \mathcal{D}_{Y} \).)

Conversely, for \( g \in \text{Hom}_{\mathcal{D}_{X}}(M, i^{0}N) \), we define \( \tilde{g} = i_{*0}M = M \otimes_{\mathcal{D}_{X}} \mathcal{D}_{X} \to N : m \otimes L \mapsto g(m)L \). Since \( \mathcal{D}_{X} \to \mathcal{D}_{Y} = \mathcal{D}_{Y} / \mathcal{J} \mathcal{D}_{Y} \), it is well-defined by the definition of \( i^{0}N \!\) ! Is it \( \mathcal{D}_{Y} \)-linear? if \( v \in \text{Vec}(Y) \), then \( \tilde{g}(m \otimes f) = g(m)Lv \). Clear! Now they are inverse to each other, just by following the definition.

By adjointness, we have a canonical morphism \( M \to i^{0}i_{0}M \) for \( M \in \mathcal{M}^{r}_{X}(\mathcal{D}_{X}) \) and \( i_{*0}i^{0}N \to N \) for \( N \in \mathcal{M}^{r}_{X}(\mathcal{D}_{Y}) \). We will show that they are isomorphisms. WLOG we may look at this locally, and by induction on codimension we may assume that \( X \) is a smooth codim 1 hypersurface defined by a single \( f \), thus locally we have an étale coordinate \( y_{1}, \ldots, y_{m} \) and corresponding vector fields \( \partial_{y_{1}}, \ldots, \partial_{y_{m}} = \partial_{f} \) on \( Y \) such that \( y_{m} = f \). We will show that \( \mathcal{D}_{X} \to \mathcal{D}_{Y} = \bigoplus_{n \in \mathbb{N}} \mathcal{D}_{X} \partial_{y}^{n} \), thus in particular free over \( \mathcal{D}_{X} \). But indeed \( \mathcal{D}_{X} \to \mathcal{D}_{Y} = \mathcal{D}_{Y} / \mathcal{J} \mathcal{D}_{Y} \), and \( \mathcal{D}_{X} = \bigoplus \mathcal{O}_{X} \partial_{y_{1}} \cdots \partial_{y_{m-1}}^{n} \).

Now, thus, \( i_{*0}M = \bigoplus_{n \in \mathbb{N}} M \partial_{y}^{n} \). What is killed by \( \mathcal{J} \), i.e. \( f \)? Since \( m \partial_{y}f = mf \partial_{y} + nm \partial_{y}^{n} = 0 \) only \( M \partial_{y}^{n} \) part is killed, thus \( i^{0}i_{0}M = M \).

Conversely, for \( N \in \mathcal{M}^{r}_{X}(\mathcal{D}_{Y}) \), \( i_{*0}N = \ker(N \xrightarrow{f} N) \). Then \( i_{*0}i^{0}N = \bigoplus_{n \in \mathbb{N}} (i^{0}N) \partial_{y}^{n} \). First note that \( i_{*0}i^{0}N \subset N \) is an injection; if \( n \partial_{y}^{n} = 0 \in i_{*0}i^{0}N \), then by applying \( f \) \( l \) times on the right we get \( n = 0 \). Also it is the direct sum since each part has different eigenvalue w.r.t. \( f \partial \). Meanwhile, we know that \( f \) acts locally nilpotently on \( N \) since \( N \) is set-theoretically supported on \( X \). Thus it suffices to show that on \( N/i_{*0}i^{0}N \) \( f \) has zero kernel. But if \( n \partial_{y} \in i_{*0}i^{0}N \), then \( n \partial_{y} = \tilde{n}f \) for some \( \tilde{n} \in i_{*0}i^{0}N \) since \( f : i_{*0}i^{0}N \to i_{*0}i^{0}N \) is a surjection. But it means \( n - \tilde{n} \in i^{0}N \subset i_{*0}i^{0}N \). Thus \( i_{*0}i^{0}N = N \) as desired. \( \square \)

## 2 Definition of \( \mathcal{M}^{r}_{X}(\mathcal{D}_{X}) \) for a Singular Variety \( X \)

Now for any affine \( X \) not necessarily nonsingular, we first choose a closed embedding \( f : X \to \mathbb{A}^{n} \), or more generally \( f : X \to Y \) where \( Y \) is a smooth affine variety and
define $M'(\mathcal{D}_X) = M'_X(\mathcal{D}_Y)$. If $X$ is smooth, then it is the same as our original definition by Kashiwara's theorem. To show that this is well-defined, you need to check that it is invariant under the choice of embedding.

**Remark.** We only need to consider the closed embedding to affine spaces, since for general $X \to Y$, then we embed $Y$ into an affine space and use Kashiwara’s theorem to deduce that $M'_X(\mathcal{D}_Y) = M'(\mathcal{D}_X)$.

Suppose we have two embeddings $i_1 : X \to \mathbb{A}^{n_1} = \mathbb{A}_1$ and $i_2 : X \to \mathbb{A}^{n_2} = \mathbb{A}_2$. Then we will construct the canonical equivalence $\phi_{21} : M_{i_1}(\mathcal{D}_X) \to M_{i_2}(\mathcal{D}_X)$ where $M_{i_1}(\mathcal{D}_X)$ are defined in the obvious sense. First, we assume $X \xrightarrow{i_1} \mathbb{A}_1 \xrightarrow{j_1} \mathbb{A}_2$ where $j$ is also a closed embedding and $j \circ i_1 = i_2$. Then we set $\phi_{21} = j_*0 : M'_X(\mathcal{D}_{\mathbb{A}_1}) \to M'_X(\mathcal{D}_{\mathbb{A}_2})$. We claim this is an equivalence; to prove this, we mimic the proof of Kashiwara’s theorem. i.e.

1. $j_*0 M'_X(\mathcal{D}_{\mathbb{A}_1}) \subset M'_X(\mathcal{D}_{\mathbb{A}_2})$?
2. define $j'^0 : M'_X(\mathcal{D}_{\mathbb{A}_2}) \to M'_X(\mathcal{D}_{\mathbb{A}_1})$? i.e. $\text{Hom}_{\mathbb{A}_2}(\mathcal{O}_{\mathbb{A}_1}, M)$ is supported on $X$?
3. how to define $\mathcal{D}_{\mathbb{A}_1}$-action?
4. are they adjoint to each other?
5. are they mutually inverses?

The only difference is that the ideal of $X$ becomes smaller, but we can still use an étale coordinate system since $\mathbb{A}_1$ and $\mathbb{A}_2$ are smooth regardless of $X$.

In general, we will use the following theorem.

**Theorem 2.1** (Schwede, [2], 3.4). Suppose $A$ and $B$ are rings. Further suppose $I$ is an ideal of $A$ and there exists a map $\gamma$ from $B$ to $A/I$. We will denote the quotient map from $A$ to $A/I$ by $\pi$. Let $X = \text{Spec}A$, $Y = \text{Spec}B$, and $Z = \text{Spec}A/I$, so that $Z$ is a closed subscheme of $X$. Then $X \cup_Z Y$ is an affine scheme with $Y$ a closed subscheme, $(X \cup_Z Y) - Y \cong X - Z$, and the maps $\alpha : X \to X \cup_Z Y$ and $\beta : Y \to X \cup_Z Y$ are morphisms of schemes.

Also following the proof, we see that if $X, Y, Z$ are affine $k$-varieties the the pushout is at least an affine scheme of finite type over $k$. In other words, in general we can always have an affine space $\mathbb{A}_{12}$ and the following commutative diagram:

```
          X \xrightarrow{i_2} \mathbb{A}_2 \\
          \downarrow{i_1} \quad \downarrow{j_2} \\
\mathbb{A}_1 \xrightarrow{j_1} \mathbb{A}_{12}
```

Now we define $\phi_{21} = (j_2)^{-1}_0 (j_1)_* : M'_X(\mathcal{D}_{\mathbb{A}_1}) \to M'_X(\mathcal{D}_{\mathbb{A}_2})$. 3
Is it well-defined? i.e. $\phi_{21}$ is canonical? for this, we use the following diagram;

\[
\begin{array}{ccc}
X & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & A_{12} \\
\downarrow & & \downarrow \\
A_3 & \rightarrow & A_{123}
\end{array}
\]

which exists by the theorem of Schwede. Since $f_{*0} : M'_X(A_{12}) \rightarrow M'_X(A''_{12})$ and $g_{*0} : M'_X(A_{12}) \rightarrow M'_X(A''_{12})$ are isomorphisms, the assertion follows from the functoriality and diagram chasing.

Now we need to show that $\phi_{21}$ satisfies the cocycle condition, from which the well-definedness of $M'(D_X)$ follows. For this, we use the same trick;

\[
\begin{array}{ccc}
A_1 & \rightarrow & A_{12} \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & A_{123} \\
\downarrow & & \downarrow \\
A_3 & \rightarrow & A_{13}
\end{array}
\]

and again this is justified by diagram chasing.

Now note that for $i : X \rightarrow \mathbb{A}^n$, we have a functor $\Gamma : M'(D_X) = M'_X(D_{\mathbb{A}^n}) \rightarrow M'(O_X) : M \mapsto \{m \in M \mid mJ = 0\} = i^!M$ as an $O_X$-module. It’s called the functor of global sections. Is it well-defined? Again using diagram chasing, we see that it is well-defined regardless of the embedding and the corresponding affine space. Thus we indeed have a functor $\Gamma : M'(D_X) \rightarrow M'(O_X)$ for an arbitrary affine variety. Note that if $X$ is smooth, then it is just ”forgetting” the $D_X$-module structure.

**Remark.** For an embedding $X \rightarrow Y$, we have $D_{X \rightarrow Y} = D_Y/JD_Y$, and by definition $\Gamma = \text{Hom}_{D_Y}(D_{X \rightarrow Y}, \bullet)$. Now if we have another embedding $j : Y \rightarrow Z$, then $j_{*0}D_{X \rightarrow Y} = D_{X \rightarrow Y} \otimes_{D_Y} D_{Y \rightarrow Z} = D_Y/JD_Y \otimes_{D_Y} D_Z/J'D_Z = O_X \otimes_{O_Y} D_Z = D_{X \rightarrow Z}$, thus using the same argument above we have a well-defined object $D_X = \{D_{X \rightarrow Y}\} \in M'(D_X)$. In this sense, $\Gamma$ is representable. Note that if $X$ is singular then $D_X$ is not an algebra and also $D(X) \neq D_X$. But it is still true that $D(X) = \text{End}(D_X)$. Now it means $D(X)$ acts on $\Gamma(M) = \text{Hom}_{D_Y}(D_X, M)$ on the right, but it does not mean that we can reconstruct $M$ by using this information. Also note that if $X$ is singular then $D_X$ may not be projective and thus $\Gamma$ may not be exact. On the smooth case, $D_X$ is projective and $\Gamma$ is exact.

### 3 $!$-crystal

If $X$ is an affine scheme of finite type (not necessarily reduced,) then we can still use the same argument to define $M'(D_X)$. Since this construction only uses the set-theoretic
support of $X$ under the embedding $X \to Y$ for a smooth $Y$, if we let $X_{red} \subset X$ be a corresponding reduced scheme, i.e. variety, then $\mathcal{M}(\mathcal{D}_X) = \mathcal{M}(\mathcal{D}_{X_{red}})$. However, $\Gamma$ is different, because for an embedding $X \to Y$, $\mathcal{D}_X = \mathcal{D}_Y / \mathcal{J} \mathcal{D}_Y$, whereas $\mathcal{D}_{X_{red}} = \mathcal{D}_Y / \sqrt{\mathcal{J}} \mathcal{D}_Y$.

It is related to the notion of $!$-crystal. Let $X$ be a variety, $M \in \mathcal{M}(\mathcal{D}_X)$, and $X'$ be an affine scheme of finite type such that $X'_{red} = X$. Then by the previous argument, $M$ defines an object in $\mathcal{M}(\mathcal{D}_{X'})$. Thus we can define $M_{X'} = \Gamma_{X'}(M)$. Now if $\eta : X' \to X''$ is a finite morphism such that its restriction on $X$ gives the identity, then by functoriality we have an isomorphism $\alpha_\eta : M_{X'} \to \eta^! M_{X''}$. (Note that $\Gamma$ is as the same as $i^!0$ if we forget the $\mathcal{D}_X$-module structure.) Also this $\alpha_\eta$ satisfies the obvious relation which is compatible with compositions of morphisms, i.e. for $Y \xrightarrow{\eta} Z \xrightarrow{\rho} W$ we have $\alpha_{\rho \eta} = (\eta^! (\alpha_\rho)) \circ \alpha_\eta$. Such a structure is called $!$-crystal on $X$. Now we have the theorem, which is as follows.

**Theorem 3.1.** The category $\text{Crys}(X)$ of $!$-crystals on $X$ is equivalent by the above functor to $\mathcal{M}(\mathcal{D}_X)$.

### 4 Push-forward for General Varieties

For $f : X \to Y$ where $X,Y$ are singular, we wish to define $f_*0$. Note that by embedding $Y$ to an affine space, we only need to deal with the case $f : X \to \mathbb{A}^n$. Now we fix an embedding $i : X \to \mathbb{A}^m$ and consider the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{A}^m \\
\downarrow{f} & & \downarrow{\theta} \\
\mathbb{A}^n
\end{array}
$$

which exists since $\mathcal{O}_{\mathbb{A}^n}$ is free. Now we define $f_*0 = \theta_*0 \circ i_*0$. Where $i_*0$ is the equivalence $\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{D}_{\mathbb{A}^m})$.

### References
