Counting points on Igusa varieties

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Abstract

Igusa varieties are smooth varieties over $\mathbb{F}_p$ which are higher-dimensional analogues of Igusa curves. They were introduced by Harris and Taylor ([HT01]) to study the bad reduction of some PEL Shimura varieties and generalized by Mantovan ([Man04], [Man05]). The present paper gives a group-theoretic formula for the traces of certain operators on the cohomology of Igusa varieties, suitable for applications via comparison with the Arthur-Selberg trace formula. Our formula generalizes the results of [HT01, V.1-V.4] to the case of any PEL Shimura varieties of type (A) and (C) and puts it in a more natural framework in the spirit of [Kot92].

1 Introduction

This article is mainly concerned with the cohomology of Igusa varieties, which are closely related to Shimura varieties. To motivate the reader, we begin with briefly discussing what has been worked out about the cohomology of Shimura varieties in relation with the Langlands correspondence.

The cohomology of Shimura varieties has been studied for decades. Apart from the case $GL_2$, the case of $U_3$ was extensively studied in [LR92]. Kottwitz and Clozel used some “simple” PEL-type Shimura varieties of unitary type (a.k.a. type (A)) and attached $n$-dimensional Galois representations to automorphic representations of $GL_n$ over CM fields satisfying the local-global compatibility of the Langlands correspondence at unramified primes. Here “simple” means that the Shimura varieties have no boundary components and that there is no endoscopy. The key inputs in their work are, among other things, the counting point formula for the good reduction fiber of the Shimura variety of type (A) or (C) ([Kot92]), to be compared with the Arthur-Selberg trace formula, and the base change result for simple unitary groups ([Clo91]). Although a stabilized version of the counting point formula was available ([Kot90]), it hinged on certain forms of the fundamental lemma, which were avoided by the use of simple unitary groups.

A new method of Harris and Taylor allows us to study the bad reduction of some simple Shimura varieties associated to the unitary (similitude) groups which arise from division algebras and are $U(1, n-1) \times U(0, n) \times \cdots \times U(0, n)$ at infinity. This had important consequences such as the proof of the local-global compatibility at ramified primes and the local Langlands conjecture for $GL_n$ over $p$-adic fields. Two main ingredients in their argument are new: the first basic identity ([HT01, Thm IV.2.7]), which was generalized by Mantovan ([Man04], [Man05]), and the second basic identity ([HT01, Thm V.5.4]), which essentially follows from the counting point formula for Igusa varieties. However, their counting point formula relies heavily on the specifics of their unitary groups and is not easily generalized.

In this work we formulate and prove a natural generalization of the counting point formula for Igusa varieties which arise from any PEL Shimura varieties of type (A) or (C). Our new formulation was inspired by the work of Kottwitz ([Kot92]) and many of his arguments indeed carry over without

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much modification. Our key observation was that reasonable definitions of the analogues of the triple \((\gamma_0; \gamma, \delta)\) and the cohomological invariant \(\alpha(\gamma_0; \gamma, \delta)\) introduced by Kottwitz can be made to work for Igusa varieties despite the apparent difference of the settings.

Our counting point formula is expected to lead to new applications regarding the computation of the cohomology of Shimura varieties and Rapoport-Zink spaces, if we combine our result with Mantovan’s formula ([Man05, Thm 22]). Thereby we may deepen our understanding of the Langlands correspondence. We will work out these applications in future writings. In many cases our formula must be stabilized to be ready for applications, as usual in the trace formula method. The stabilization will be provided in the sequel paper ([Shi]). We merely remark that we do not need the twisted fundamental lemma for stabilization since our formula does not involve twisted orbital integrals. (Unlike in the case of Kottwitz’s formula; see [Kot90, p.180].) We also add that the cohomology of Rapoport-Zink spaces was computed by Fargues ([Far04]) in several cases where the Igusa varieties are zero-dimensional, using a version of Mantovan’s formula (which is simpler in those cases) and techniques from rigid analytic geometry, among others. In the cases considered by Fargues, the issue of stabilization does not arise.

We briefly explain the structure of this article. In §2-4 we build up background materials. The readers may skip this part and come back later for references. The main discussion begins in §5 where we construct a Shimura variety \(X\) defined over the reflex field \(E\) along with an integral model, starting from an integral Shimura PEL datum \((B, \mathcal{O}_B, \ast, V, \Lambda_0, \langle \cdot, \cdot \rangle, h)\) of type (A) or (C). Apart from the assumptions ensuring that \(G\) is unramified at \(p\), we do not make further restrictions. In particular \(X\) need not be proper over \(E\). Denote by \(G\) the associated algebraic group over \(\mathbb{Q}\). Let \(J_0\) be the \(\mathbb{Q}_p\)-group arising as the automorphism group of an isocrystal of type \(h\), which turns out to be an inner form of a Levi subgroup of \(G_{\mathbb{Q}_p}\). For each Newton polygon datum \(b \in B(G, -\mu_b)\), we define the Igusa variety \(\text{Ig}_b\) as a projective system. It is worth noting that \(G(\mathbb{A}^\infty_p) \times J_0(\mathbb{Q}_p)\) acts on \(H_c(\text{Ig}_b, \mathcal{L}_\xi)\) while \(G(\mathbb{A}^\infty) \times \text{Gal}(\overline{E}/E)\) acts on \(H(X, \mathcal{L}_\xi)\), where \(\mathcal{L}_\xi\) is an \(l\)-adic sheaf constructed from an algebraic representation of \(G\).

From §6 until the end is devoted to obtaining the following main result, namely the counting point formula for the cohomology of Igusa varieties.

**Theorem 1** (Theorem 13.1). If \(\varphi \in C_c^\infty(G(\mathbb{A}^\infty_p) \times J_0(\mathbb{Q}_p))\) is acceptable, then
\[
\text{tr} \left( \varphi \big| H_c(\text{Ig}_b, \mathcal{L}_\xi) \right) = \sum_{(\gamma_0; \gamma, \delta) \in KT_0} \text{vol}(I_\infty(\mathbb{R}))^{-1} |\text{tr}(\gamma_0)| \cdot \text{tr} \left( \xi(\gamma_0) \circ G(\mathbb{A}^\infty_p) \times J_0(\mathbb{Q}_p) \right) (\varphi)
\]

The proof is done in several steps. We use Fujiwara’s trace formula in §6 to convert the computation of \(\text{tr} \left( \varphi \big| H_c(\text{Ig}_b, \mathcal{L}_\xi) \right)\) into the problem of counting \(\overline{\mathbb{F}}_p\)-points of \(\text{Ig}_b\) fixed under correspondences. We define the notion of acceptable functions so that this works whenever \(\varphi\) is acceptable. In §7 using the moduli interpretation of \(\text{Ig}_b\), the counting point problem is essentially reduced to parametrizing the triples \((A, \lambda, i)\) that appear in the moduli data and conjugacy classes \([a]\) in the automorphism group of \((A, \lambda, i)\) in the isogeny category. We carry out this parametrization in terms of Kottwitz triples \((\gamma_0; \gamma, \delta)\) which have a purely group-theoretic description.

Useful lemmas for studying \((A, i)\) and \(\lambda\) are provided in §8 and §9, respectively, using tools from the Honda-Tate theory and Galois cohomology. We remark that Lemma 8.6 looks simple but is important in working with conjugacy classes \([a]\). In §10 and §11 we give the definition of Kottwitz triples \((\gamma_0; \gamma, \delta)\) and an important result on the vanishing of the cohomological invariant (Corollary 11.3). In §12 we complete the proof of the reparametrization of \((A, \lambda, i)\) and \([a]\) in terms of \((\gamma_0; \gamma, \delta)\). In going forward the deepest fact seems the vanishing of \(\alpha(\gamma_0; \gamma, \delta)\), which is a direct consequence of Corollary 11.3. In going backward, we recover \((A, i)\) via Honda-Tate theory by reading off the
necessary data from \((\gamma_0; \gamma, \delta)\). The rationality of \(\lambda\) and \(a\) is proved in two steps and follows from the vanishing of \(\alpha(\gamma_0; \gamma, \delta)\). At this point it is easy to deduce the main result, namely Theorem 13.1.

It is worth noting that [a] did not show up in the work of Kottwitz. In some sense the conjugacy classes [a] reflect the level structure at \(p\) of Igusa varieties and add one more layer to the whole argument. Among the cohomological invariants in §10 \(\beta(\gamma_0; \gamma, \delta)\) is a direct analogue of \(\alpha(\gamma_0; \gamma, \delta)\) defined by [Kot92] and encodes the rationality of \((A, \lambda, i)\) whereas our \(\alpha(\gamma_0; \gamma, \delta)\) encodes the rationality of \((A, \lambda, i)\) and the conjugacy class [a]. As such \(\beta(\gamma_0; \gamma, \delta)\) plays only an auxiliary role in the proof of Lemma 12.3.

Finally we mention that this work is largely based on Chapter 1-3 of the author’s Harvard thesis ([Shi07]) but made more focused on the goal of establishing the counting point formula. We also corrected minor errors in [Shi07] and changed some convention to be more compatible with the literature. We refer to [Shi07] only once, in the proof of Lemma 6.3, for minor details which are not difficult but somewhat distracting.

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Notation

First of all, we point out the notations different from those of [Man05]. We write \(J_0\) and \(I_{gs}\) for the \(p\)-adic group \(T_0\) and the Igusa variety \(J_0\) of that paper, respectively. The notation \(J_0\) seems to have been widely used (for instance in [RZ96]).

A CM field is by definition an imaginary quadratic extension of a totally real field. A CM field has a well-defined automorphism of order 2, which is a restriction of the complex conjugation \(\sigma\).

Now suppose that \(B\) is a finite dimensional \(\mathbb{Q}\)-semisimple algebra. For a semisimple element \(\gamma \in B^\times\), define \(F(\gamma)\) to be the commutative \(F\)-subalgebra generated by \(\gamma\). For a \(\mathbb{Q}\)-algebra \(R\), we often write \(B_R\) or \(B \otimes R\) for \(B \otimes_{\mathbb{Q}} R\). If a tensor product is taken over anything other than \(\mathbb{Q}\), the base ring will be written out explicitly.

For a number field \(F\), we define the following notation. When \(v\) is a place of \(F\), denote by \(F_v\) the completion of \(F\) with respect to the metric defined by \(v\). Write \(\varpi_v\) for a uniformizer of the integer ring \(O_{F_v}\) of \(F_v\). The residue field \(O_{F_v}/(\varpi_v)\) is denoted \(k(v)\). When \(S\) is a finite set of places of \(F\) define \(\mathcal{A}_F^S\) to be the restricted product \(\prod_{v \in S} F_v\). Define \(\mathcal{A}_F^S := \text{lim} \mathcal{A}_F^{S'}\) where \(S'\) runs over finite extension fields over \(F\) and \(S(F')\) denotes the set of places of \(F'\) over \(S\). If \(F = \mathbb{Q}\) we simply write \(\mathcal{A}_F^S\) and \(\mathcal{A}_F^S\).

Now suppose that \(G\) is a connected reductive group defined over a field \(F\). For \(g \in G(F)\), let \(\text{Int}(g)\) denote the inner automorphism \(x \mapsto gxg^{-1}\) of \(G\) and \(\text{Int}(G)\) the group of inner automorphisms of \(G\). We write \(H^1(F, G)\) for \(H^1(\text{Gal}(F^{\text{sep}}/F), G(F^{\text{sep}}))\) where \(F^{\text{sep}}\) is a separable closure of \(F\). The dual group of \(G\) will be written as \(\hat{G}\). It is a complex Lie group with \(\text{Gal}(F^{\text{sep}}/F)\) action. When \(F\) is a number field, we define \(\text{ker}^v(F, G) := \ker(H^1(F, G) \to \bigoplus_v H^1(F_v, G))\) where \(v\) runs over all places of \(F\).

The notation \(Z(A)\) (resp. \(Z_A(a)\)) will be used to denote the center (resp. the centralizer of \(a\)) in \(A\) where \(A\) is either a group, an algebra, or an algebraic group.

We use symbols \(\Gamma\) and \(\Gamma(v)\) to mean \(\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) and \(\Gamma(v) = \text{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)\) starting from §5. Here \(v\) can be any place of \(\mathbb{Q}\), including the infinite place \(\infty\).
When $G$ is a topological group, $C_{\infty}^*(G)$ denotes the space of locally constant compactly supported functions with values in a fixed characteristic 0 field, which is often $\mathbb{Q}_p$. (In this paper $G$ will be a $p$-adic Lie group or a restricted product of such.)

We use the notation Groth($\cdot$) for the Grothendieck group of admissible representations of topological groups. For precise definition, refer to [HT01, I.2].

## 2 Hermitian modules

Let $C$ be a finite dimensional $\mathbb{Q}$-algebra with an involution $\ast$. For any $\mathbb{Q}$-algebra $A$, the involution $\ast$ extends to $C \otimes \mathbb{Q} A$, acting as the identity on $A$.

**Definition 2.1.** Consider a finite free $C \otimes \mathbb{Q} A$-module $V$ equipped with a non-degenerate $A$-bilinear pairing $\langle \cdot, \cdot \rangle : V \times V \to A$. We say that $(V, \langle \cdot, \cdot \rangle)$ is a $\ast$-Hermitian $C \otimes \mathbb{Q} A$-module if

$$\langle \gamma x, y \rangle = \langle x, \gamma^* y \rangle \quad \text{for all } x, y \in V \text{ and all } \gamma \in C \otimes \mathbb{Q} A.$$

In this case, $\langle \cdot, \cdot \rangle$ is called a $\ast$-Hermitian pairing (with respect to $C \otimes \mathbb{Q} A$).

When $\langle \cdot, \cdot \rangle$ and $\ast$ are understood, we simply say that $V$ is a Hermitian $C \otimes \mathbb{Q} A$-module. Two $\ast$-Hermitian $C \otimes \mathbb{Q} A$-modules $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ are said to be equivalent if there exist an isomorphism of $C \otimes \mathbb{Q} A$-modules $\delta : V_1 \rightarrow V_2$ and an element $\mu \in A^\times$ such that $\langle x, y \rangle_1 = \mu \langle \delta x, \delta y \rangle_2$ for all $x, y \in V_1$.

Given a $\ast$-Hermitian pairing $\langle \cdot, \cdot \rangle_0$ on $V$ with respect to $C \otimes \mathbb{Q} A$, define an algebraic group $H$ over Spec $A$ consisting of self-equivalences by

$$H(A) = \{ h \in \text{End}_{C \otimes \mathbb{Q} A}(V \otimes \mathbb{Q} A) \mid \exists \varpi(h) \in A^\times, \langle hw_1, hw_2 \rangle_0 = \varpi(h) \langle v_1, v_2 \rangle_0 \text{ for all } v_1, v_2 \in V \otimes \mathbb{Q} A \}$$

for any $\mathbb{Q}$-algebra $A$.

Let $F$ be a field of characteristic 0. Fix a $\ast$-Hermitian $C \otimes \mathbb{Q} F$-module $V$. Define $St(V)$ to be the set of equivalence classes of $\ast$-Hermitian $C \otimes \mathbb{Q} F$-modules $W$ which are isomorphic to $V$ as $C \otimes \mathbb{Q} F$-modules (without pairing). We view $St(V)$ as a pointed set where the equivalence class of $V$ is distinguished. It is possible to construct a natural map $St(V) \rightarrow H^1(F, H)$ as follows. For $W \in St(V)$, choose an equivalence $h : V \otimes_F \overline{F} \simeq W \otimes_F \overline{F}$ as a $C \otimes \mathbb{Q} \overline{F}$-Hermitian modules. (Use the fact that any two $C \otimes \mathbb{Q} \overline{F}$-Hermitian pairings on a $C \otimes \mathbb{Q} \overline{F}$-module are equivalent.) Then $h^{-1} h^\sigma \in H(\overline{F})$ where $h^\sigma = (1 \otimes \sigma) h (1 \otimes \sigma)^{-1}$ for $\sigma \in \text{Gal}(\overline{F}/F)$. The cocyle $\sigma \mapsto h^{-1} h^\sigma$ is the desired element of $H^1(F, H)$ associated to $W$. The following lemma is easy.

**Lemma 2.2.** The above map defines a natural isomorphism of pointed sets between $St(V)$ and $H^1(F, H)$.

Now let $F$ be a number field and $v$ denote a place of $F$. Let

$$\Gamma := \text{Gal}(\overline{F}/F) \quad \text{and} \quad \Gamma(v) := \text{Gal}(\overline{F}_v/F_v).$$

Let $G$ be a connected reductive group over $F$. We define

$$A(G) := \pi_0(\mathbb{Z}(\widehat{G})^D), \quad A_v(G) := \pi_0(\mathbb{Z}(\widehat{G})^{\Gamma(v)})^D$$

where $D$ means the Pontryagin dual. Note that there is a natural restriction map $A_v(G) \rightarrow A(G)$. In terms of the algebraic fundamental group $\pi_1(G)$, we have canonical isomorphisms $A(G) \simeq (\pi_1(G)_{\Gamma})_{\text{tor}}$ and $A_v(G) \simeq (\pi_1(G)_{\Gamma(v)})_{\text{tor}}$ ([Bor98, Prop. 1.10]). (The subscript "tor" denotes the torsion subgroup therein.) Since the construction of $\pi_1(G)$ is functorial in $G$ with respect to any $F$-morphism of connected reductive groups over $F$, it is easy to see that the groups $A_v(G)$, $A(G)$ and the map $A_v(G) \rightarrow A(G)$ are functorial in $G$. 


Lemma 2.3. For every place $v$ of $F$, there is a canonical map
\[ \alpha_{G,v} : H^1(F_v, G) \to A_v(G) \]
which is an isomorphism if $v$ is a finite place. This map $\alpha_{G,v}$ is functorial in $G$ (with respect to any $F$-morphism). When composed with the natural map $A_v(G) \to A(G)$, the maps $\alpha_{G,v}$ induce a map $H^1(F, G(\overline{K}_F)) \to A(G)$ fitting into an exact sequence
\[ 1 \to \ker^1(F, G) \to H^1(F, G) \to H^1(F, G(\overline{K}_F)) \to A(G) \]

Proof. The lemma is proved in [Kot86, Thm 1.2, Prop 2.6] except that the functoriality of $\alpha_{G,v}$ is proved only for normal morphisms of reductive groups over $F$. (Some more cases are covered in Lemma 4.3 of that paper.) However functoriality is easily extended to all cases. When $G_{der}$ is simply connected, write $D_G := G/G_{der}$. Recall from the same paper that $\alpha_{G,v}$ is the same as the composite map
\[ H^1(F_v, G) \to H^1(F_v, D_G) \cong A_v(D_G) = A_v(G) \cong (\pi_1(G)_{\Gamma(v)})_{tor} \]
where the isomorphisms are canonical. From this it is easy to prove functoriality with respect to any $F$-morphism $G_1 \to G_2$ granted that $G_{1,der}$ and $G_{2,der}$ are simply connected. The general case of functoriality is proved as in [Kot86, p.369] using $z$-extensions.

Alternatively, functoriality can be established in full generality using the following canonical functorial maps ([Lab99, Prop 1.6.7, Prop 1.7.3], also [Bor98, Cor 5.5]) in the context of abelianized cohomology
\[ H^1(F_v, G) \to H^1_{ab}(F_v, G) \to A_v(G) \cong (\pi_1(G)_{\Gamma(v)})_{tor}, \]
whose composite is $\alpha_{G,v}$. (Of course the surjection and the injection above are isomorphisms when $v$ is non-archimedean.) \qed

In the statement of the lemma, the map $\alpha_{G,v}$ is canonical in the sense that it is uniquely determined by two conditions: (i) $\alpha_{G,v}$ is the canonical map induced by Tate-Nakayama duality when $G$ is a torus and (ii) $\alpha_{G,v}$ is functorial in $G$. The meaning of canonicality for $\beta_G$ is taken to be the same in the following lemma.

Lemma 2.4. There is a canonical map $\beta_G : H^1(F, G(\overline{K}_F)/Z(G)(\overline{F})) \to A(G)$ which is functorial in $G$. When $\beta_G$ is composed with the natural map $H^1(F, G(\overline{K}_F)) \to H^1(F, G(\overline{K}_F)/Z(G)(\overline{F}))$, the resulting map $H^1(F, G(\overline{K}_F)) \to A(G)$ is identical to the map induced by $\alpha_{G,v}$. Moreover, the map $\beta_G$ fits into an exact sequence
\[ 1 \to H^1(F, G(\overline{K}_F)/Z(G)(\overline{F})) \to H^1(F, G(\overline{K}_F)/Z(G)(\overline{F})) \to A(G) \]

Proof. Everything in the lemma is proved in [Kot86, Thm 2.2, Cor 2.5] except that the functoriality of $\beta_G$ is verified only for normal morphisms. The general case of functoriality is proved as in the first paragraph of the proof of Lemma 2.3, noting that ([Kot86, p.374]) if $G_{der}$ is simply connected, $\beta_G$ is the composition
\[ H^1(F, G(\overline{K}_F)/Z(G)(\overline{F})) \to H^1(F, D_G(\overline{K}_F)/D_G(\overline{F})) \to A(D) = A(G). \]
\qed

From here until the end of this section, let $(B, *, V, \langle \cdot, \cdot \rangle)$ be a partial Shimura datum in Definition 5.1. By linearly extending $\langle \cdot, \cdot \rangle$, we have a $*$-Hermitian $B_{Q_*,*}$-module $V_{Q_*,*}$ for each place $v$ of $Q$ and a $*$-Hermitian $B \otimes Q A^{S}$-module $V \otimes Q A^{S}$. Here $A^{S}$ is the ring of adeles with trivial entries at the places
contained in the finite set $S$. Note that we will often write $B_{Q_v}$ and $V_{Q_v}$ for $B \otimes_{Q_v} Q$ and $V \otimes_{Q_v} Q_v$. By Lemma 2.2, we have horizontal bijections in the following commutative diagram.

$$
\begin{array}{ccc}
H^1(Q, G) & \xrightarrow{1-1} & St(V) \\
\downarrow & & \downarrow \\
H^1(Q_v, G) & \xrightarrow{1-1} & St(V_{Q_v})
\end{array}
$$

3 Conjugacy classes and Galois cohomology

In this subsection, we summarize various results concerning conjugacy classes and Galois cohomology of reductive groups from [Kot84b] and [Kot86]. We assume the reader is familiar with the dual groups and $L$-groups, for which one can see [Bor79].

Let $F$ be a perfect field. Let $\overline{F}$ be an algebraic closure of $F$. Let $G$ be a connected reductive group over $F$ and assume that its derived subgroup $G^{der}$ is simply connected.

**Definition 3.1.** We say that $\gamma, \gamma' \in G(F)$ are ($F$-)conjugate, or stably conjugate if $\gamma' = g\gamma g^{-1}$ for some $g \in G(F)$ or $G(\overline{F})$, respectively. Write $\gamma \sim \gamma'$ or $\gamma \sim_{st} \gamma'$ (equivalently $\gamma \sim_{\Gamma} \gamma'$) in each case.

By our assumption that $G^{der}$ is simply connected, $I := Z_G(\gamma)$ is a connected reductive group over $F$. As there is a canonical $\Gamma$-equivariant embedding $Z(\hat{G}) \hookrightarrow Z(\hat{I})$, we may consider the exact sequence of $\Gamma$-modules

$$1 \to Z(\hat{G}) \to Z(\hat{I}) \to Z(\hat{I})/Z(\hat{G}) \to 1$$

which gives us a long exact sequence ([Kot84b, Cor 2.3]), part of which is

$$X_*(Z(\hat{I})/Z(\hat{G}))^\Gamma \to \pi_0(Z(\hat{G})) \to \pi_0(Z(\hat{I})^\Gamma) \to \pi_0((Z(\hat{I})/Z(\hat{G}))^\Gamma) \xrightarrow{\xi} H^1(F, Z(\hat{G})) \to H^1(F, Z(\hat{I}))$$

At each place $v$ of $F$, we have a similar sequence and in particular a homomorphism $\xi_v: \pi_0((Z(\hat{I})/Z(\hat{G}))^\Gamma(v)) \to H^1(F_v, Z(\hat{G}))$. We define

$$\mathcal{R}(I/F) := \xi^{-1}(\ker H^1(F, Z(\hat{G}))) \quad \text{and} \quad \mathcal{R}(I/F_v) := \ker \xi_v.$$ 

Since the following diagram commutes with the horizontal maps being obvious ones, we have a canonical map $\mathcal{R}(I/F) \to \mathcal{R}(I/F_v)$.

$$
\begin{array}{ccc}
\pi_0((Z(\hat{I})/Z(\hat{G}))^\Gamma) & \xrightarrow{\xi} & \pi_0((Z(\hat{I})/Z(\hat{G}))^\Gamma(v)) \\
\downarrow & & \downarrow \\
H^1(F, Z(\hat{G})) & \xrightarrow{\xi_v} & H^1(F_v, Z(\hat{G}))
\end{array}
$$
From the definition of $\mathfrak{R}(I/F)$ and $\mathfrak{R}(I/F_v)$ the following exact sequences are immediate.

$$\pi_0(Z(\widehat{G})) \rightarrow \pi_0(Z(\widehat{I})) \rightarrow \mathfrak{R}(I/F) \rightarrow \ker^1(F,Z(\widehat{G})) \rightarrow \ker^1(F,Z(\widehat{I}))$$  \hspace{1cm} (3)

$$\pi_0(Z(\widehat{G}))^{F(v)} \rightarrow \pi_0(Z(\widehat{I}))^{F(v)} \rightarrow \mathfrak{R}(I/F_v) \rightarrow 1$$  \hspace{1cm} (4)

It is well known that the last arrow in (3) is an isomorphism if $G$ is an algebraic group arising from the PEL-type moduli problem of Shimura varieties of type $A$ or $C$ (use [Kot92, §7]). We remark that if $\gamma$ is elliptic in $G(F)$ (resp. in $G(F_v)$), then the first arrow in (3) (resp. (4)) is injective since the ellipticity means that $X_*(Z(\widehat{I})/Z(\widehat{G}))^F$ (resp. $X_*(Z(\widehat{I})/Z(\widehat{G}))^{F(v)}$) is trivial.

Suppose that semisimple elements $\gamma_v$ and $\gamma'_v$ of $G(F_v)$ are stably conjugate to each other. Lemma 2.3 implies that we have a canonical map

$$\ker(H^1(F_v, I) \rightarrow H^1(F_v, G)) \rightarrow \ker(A_v(I) \rightarrow A_v(G)) \simeq \mathfrak{R}(I/F_v)$$

where the last natural isomorphism comes from the dual of the sequence (4). Choose an element $g \in G(F_v)$ such that $\gamma' = g\gamma g^{-1}$. We denote by $\text{inv}_v(\gamma_v, \gamma'_v)$ the image of the 1-cocycle $\sigma \mapsto g^{-1}og$ in $\mathfrak{R}(I/F_v)$ above the map. By abuse of notation, $\text{inv}_v(\gamma_v, \gamma'_v)$ will also be viewed as an element of $\mathfrak{R}(I/F)\mathfrak{D}$ via the canonical map $\mathfrak{R}(I/F) \rightarrow \mathfrak{R}(I/F_v)$.

Let $\gamma = (\gamma_v)$ and $\gamma' = (\gamma'_v)$ be semisimple elements of $G(\mathbb{A}_F)$ that are $\mathbb{A}_F$-conjugate to each other. We define the following element of $\mathfrak{R}(I/F)\mathfrak{D}$

$$\text{inv}(\gamma, \gamma') := \sum_v \text{inv}_v(\gamma_v, \gamma'_v).$$

The following is an important result regarding rationality of conjugacy classes.

**Lemma 3.2.** ([Kot86, Thm 6.6]) Suppose that two semisimple elements $\gamma \in G(F)$ and $\gamma' \in G(\mathbb{A}_F)$ are conjugate in $G(\mathbb{A}_F)$. The element $\gamma' \in G(\mathbb{A}_F)$ is $G(\mathbb{A}_F)$-conjugate to an element of $G(F)$ if and only if $\text{inv}(\gamma, \gamma')$ is trivial.

We can relate the conjugacy classes to Hermitian modules. Suppose that $(B,*,V,\langle \cdot,\cdot \rangle)$ comes from a partial Shimura PEL datum and $G$ is the associated group ($\S 5$). Put $F := Z(B)$. Each semisimple element $\gamma \in G(\mathbb{Q})$ generates a $F$-subalgebra $F(\gamma)$ in $B$ and naturally induces a Hermitian $B \otimes_F F(\gamma)$-module structure on $V$, which we call $V_\gamma$. Let us write $\text{St}(\gamma,V)$ for the set of equivalence classes of Hermitian $B \otimes_F F(\gamma)$-modules which are equivalent to $V$ as Hermitian $B$-modules. $\text{St}(\gamma,V)$ is naturally a pointed set with the equivalence class of $V_\gamma$ distinguished. From the definition, we have $\text{St}(\gamma,V) = \ker(\text{St}(V_\gamma) \rightarrow \text{St}(V))$. A local analogue $\text{St}(\gamma_v,V_{q_\ell})$ is defined in the exactly same way.

Recall that we earlier had an isomorphism of pointed sets $\text{St}(G(\mathbb{Q}),\gamma) \simeq \ker(H^1(\mathbb{Q}, Z_G(\gamma)) \rightarrow H^1(\mathbb{Q}, G))$. Lemma 2.2 implies that there is a natural map $\text{St}(\gamma, V) \rightarrow \ker(H^1(\mathbb{Q}, Z_G(\gamma)) \rightarrow H^1(\mathbb{Q}, G))$. There is also a natural map $\text{St}(G(\mathbb{Q}),\gamma) \rightarrow \text{St}(\gamma, V)$, which we now describe. The element $\gamma$ maps to the Hermitian $B \otimes_F F(\gamma)$-module $V_\gamma$. If $\gamma' = g\gamma g^{-1}$ for $g \in G(\mathbb{Q})$, then $\gamma'$ endows $V$ with a Hermitian $B \otimes_F F(\gamma')$-module structure, which we call $V_{\gamma'}$. As $\gamma \mapsto \gamma'$ induces an isomorphism $F(\gamma) \simeq F(\gamma')$, we see that $V_{\gamma'}$ is naturally an element of $\text{St}(\gamma, V)$. The map $\text{St}(G(\mathbb{Q}),\gamma) \rightarrow \text{St}(\gamma, V)$ given by $\gamma' \mapsto V_{\gamma'}$ is well-defined.

**Lemma 3.3.** For $\gamma \in G(\mathbb{Q})$, the three pointed sets $\text{St}(G(\mathbb{Q}),\gamma)$, $\text{St}(\gamma, V)$ and $\ker(H^1(\mathbb{Q}, Z_G(\gamma)) \rightarrow H^1(\mathbb{Q}, G))$ are isomorphic to each other via the natural maps defined above. Moreover, these maps form a commutative diagram.

\[
\begin{array}{ccc}
\text{St}(G(\mathbb{Q}),\gamma) & \longrightarrow & \text{St}(\gamma, V) \\
\ker(H^1(\mathbb{Q}, Z_G(\gamma)) \rightarrow H^1(\mathbb{Q}, G)) & \longmapsto & \end{array}
\]
For $\gamma_v \in G(\mathbb{Q}_v)$ there is a similar commutative diagram of isomorphisms among $\text{St}_{G(\mathbb{Q}_v)}(\gamma_v)$, $\text{St}(\gamma_v, V_{\mathbb{Q}_v})$ and $\ker(H^1(\mathbb{Q}_v, Z_G(\gamma_v))) \to H^1(\mathbb{Q}_v, G))$.

Proof. Immediate from the construction of maps. \hfill \square

4 Isocrystals and Barsotti-Tate groups with additional structure

Here we study isocrystals with additional structure following mainly [RR96], [Kot85] and [Kot97]. We mostly keep their convention but note that some other conventions are also used in the literature.

It is worth noting that a Barsotti-Tate group over $\mathbb{Q}_p$ of pure slope $\lambda$ corresponds to an isocrystal of pure slope $-\lambda$ under our covariant Dieudonné functor $\mathcal{V}$ introduced later in this section. Because of this the elements $b \in B(G)$ parametrizing the Newton polygon strata of Shimura varieties will not lie in $B(G, \mu)$ but in $B(G, -\mu)$. (See §5 and also Example 4.3).

We set up the notation for this section.

- $W := W(\overline{\mathbb{F}}_p)$ is the ring of Witt vectors.
- $L$ is the fraction field of $W$.
- $\Gamma := \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (In later sections $\Gamma$ usually denotes $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)
- $G$ is a connected reductive group over $\mathbb{Q}_p$.
- $T \subset G$ is a maximal torus defined over $\mathbb{Q}_p$, with Weyl group $\Omega$.
- $A$ is a maximal $\mathbb{Q}_p$-split torus with Weyl group $\Omega_{\mathbb{Q}_p}$.
- $\mathcal{D}$ is the pro-algebraic torus with character group $\mathcal{Q}$.
- $\sigma$ is the Frobenius element in $\text{Gal}(L/\mathbb{Q}_p)$ inducing $x \mapsto x^p$ on the residue field.
- $L_s$ is the fixed field of $L$ under $\sigma^s$ $(s > 0)$.

We introduce two set-valued functors on the category of connected reductive groups over $\mathbb{Q}_p$.

- $B(G) = G(L)/\sim$, $x \sim y \Leftrightarrow \exists g \in G(L), x = g^{-1}yg^p$
- $N(G) = (\text{Int} \ G(L) \backslash \text{Hom}_L(\mathcal{D}, G))^{(\sigma)} \simeq (X_*(T)_{\mathbb{Q}}/\Omega)^{\sigma} \simeq X_*(A)_{\mathbb{Q}}/\Omega_{\mathbb{Q}_p}$

The last isomorphism is (1.1.3.1) of [Kot84a]. For each connected reductive group $G$ over $\mathbb{Q}_p$, there is a map

$$\nu_G : G(L) \to \text{Hom}_L(\mathcal{D}, G)$$

characterized by various properties ([RR96, Thm 1.8]). Moreover, $\nu_G$ induces a natural transformation of functors $\nu : B(\cdot) \to N(\cdot)$, yielding the Newton map $\nu_G : B(G) \to N(G)$ for each connected reductive group $G$.

We have the following commutative diagram which is functorial in $G$. (Cf. [RR96, 1.15], noting that $\pi_1(G) = X^*(Z(\mathcal{G}))$. The first row is exact as pointed sets and the second row is exact as abelian groups. The map $\delta_G$ and the horizontal arrow in the lower right corner are explained in [RR96]. The left top arrow is basically sending a cocycle in $H^1(\mathbb{Q}_p, G)$ to its evaluation at $\sigma$. (See [Kot85, 1.8] for careful definition.)

$$\begin{array}{ccc}
H^1(\mathbb{Q}_p, G) & \longrightarrow & B(G) \xrightarrow{\nu_G} N(G) \\
\downarrow{\kappa_G} & & \downarrow{\kappa_G} \\
A_p(G) & \longrightarrow & X^*(Z(\mathcal{G}))^{\Gamma} \longrightarrow X^*(Z(\mathcal{G}))_{\Gamma} \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}$$

\hfill (5)
Given a cocharacter $\mu \in X_*(T) = X^*(\hat{T})$, the finite subset $B(G, \mu)$ of $B(G)$ is defined in [Kot97, §6] (cf. Example 4.3). Let $\mu_1 \in X^*(Z(\hat{G})^\Gamma)$ be the restriction of $\mu$. Then every element in $B(G, \mu)$ maps to $\mu_1$ under $s_G$.

**Definition 4.1.** An element $\tilde{b} \in G(L)$ is called decent if for some $s \in \mathbb{Z}_{>0}$, $s\nu_G(\tilde{b})$ arises from a genuine morphism $G_m \to G$ and

$$\tilde{b} \sigma(\tilde{b}) \cdots \sigma^{s-1}(\tilde{b}) = s\nu_G(\tilde{b})(\mu). \quad (6)$$

From here until the end of the current section, assume that $G$ is quasi-split over $\mathbb{Q}_p$. Given $b \in B(G)$, choose a decent representative $\tilde{b} \in G(L)$ of $b$, which is always possible by [Kot85, 4.3]. In fact, we can choose $\tilde{b}$ such that the centralizer of $\nu_G(\tilde{b})$ is defined over $\mathbb{Q}_p$ ([Kot85, p.219]). Write $M_\delta$ for this Levi subgroup of $G$. On the other hand, define an algebraic group $J_\delta$ over $\mathbb{Q}_p$ by the relation

$$J_\delta(R) = \{ g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g = \tilde{b} \sigma(g) \tilde{b}^{-1} \}$$

for any $\mathbb{Q}_p$-algebra $R$. The group functor $J_\delta$ is shown to be representable in [RZ96, 1.12].

For different representatives $\tilde{b}$, the $\mathbb{Q}_p$-groups $J_\delta$ are canonically isomorphic to each other over $\mathbb{Q}_p$. For any two $\tilde{b}$ such that $M_\delta$ is defined over $\mathbb{Q}_p$, the pairs $(\tilde{b}, M_\delta)$ are conjugate to each other by an element of $G(\mathbb{Q}_p)$ ([Kot85, Prop 6.3]). For future convenience we may and will arrange that $\tilde{b}$ is a decent element of $M_\delta(L)$ (not just $G(L)$) using [Kot85, Prop 6.2]. In practice, we will write $J_\delta$ (resp. $M_\delta$) for $J_\delta$ (resp. $M_\delta$) by agreeing that a choice of a decent representative $\tilde{b}$ in the $\sigma$-conjugacy class $b$ will be fixed. We remark that the fibers of the map $\nu_G$ can be described using $J_\delta$. For each $b \in B(G)$, the set $\{b' \in B(G) \mid \nu_G(b') = \nu_G(b) \}$ is a principal homogeneous space for $H^1(\mathbb{Q}_p, J_\delta)$. (See [RR96, Prop 1.17].)

**Lemma 4.2.** If $\tilde{b} \in G(L)$ is decent for $s \in \mathbb{Z}_{>0}$, then $\tilde{b}$ belongs to $G(L_s)$ and there is an isomorphism $J_\delta \simeq M_\delta$ over $L_s$ by which $J_\delta$ is an inner form of $M_\delta$ over $\mathbb{Q}_p$. In case $\tilde{b} \in M_\delta(L)$, this inner form is represented by the cocycle $\sigma \mapsto \text{Int}(\tilde{b})$ in $H^1(L_s/\mathbb{Q}_p, \text{Int}(M_\delta))$.

**Proof.** Corollary 1.9 and 1.14 (and the proof for the latter) in [RZ96].

It is easy to see that the embedding $J_\delta \times_{\mathbb{Q}_p} \mathbb{Q}_p \hookrightarrow G \times_{\mathbb{Q}_p} \mathbb{Q}_p$ given by $J_\delta \times_{\mathbb{Q}_p} \mathbb{Q}_p \simeq M_\delta \times_{\mathbb{Q}_p} \mathbb{Q}_p$ and the natural embedding $M_\delta \hookrightarrow G$ is canonical up to $G(\mathbb{Q}_p)$-conjugacy.

**Example 4.3.** Consider the case $G = \text{Res}_{K/\mathbb{Q}_p} GL_n$ where $K$ is a finite extension of $\mathbb{Q}_p$.

First observe that $N(G) \simeq (\mathbb{Q}^n)^{S_n}$, where $S_n$ denotes the symmetry group of $n$ variables, can be identified with the set of the following data:

$$(r, \{\lambda_i\}_{1 \leq i \leq r}, \{m_i\}_{1 \leq i \leq r}) \text{ such that } \lambda_i \in \mathbb{Q}, r, m_i \in \mathbb{Z}_{>0}, \lambda_1 < \cdots < \lambda_r, \sum_{i=1}^r m_i = n.$$  

We exhibit $B(G)$ as the image of the map $\nu_G : B(G) \to N(G)$, which turns out to be injective. Each image $\nu_G(b)$ is given by rational numbers $\lambda_1 < \cdots < \lambda_r$ and the multiplicities $m_i$ of $\lambda_i$. Our normalization is that if $G = GL_1$ then $\nu_G$ sends the uniformizer of $K$ to 1. The image of $\nu_G$ is characterized by the condition

$$\forall i, m_i \lambda_i \in \mathbb{Z}. \quad (7)$$
Suppose that $\bar{\nu}_G(b) = (r, \{\lambda_i\}, \{m_i\})$. Then $\delta_G(r, \{\lambda_i\}, \{m_i\}) = \sum_i \lambda_i m_i$, $\kappa_G(b) = \sum_i \lambda_i m_i$ and

$$J_b = \text{Res}_{\mathbb{Q}/\mathbb{Q}_p} \prod_{i=1}^r GL_{m_i/h_i}(D - \lambda_i), \quad M_b = \text{Res}_{\mathbb{Q}/\mathbb{Q}_p} \prod_{i=1}^r GL_{m_i}. \quad (8)$$

where $D - \lambda_i$ denotes the division algebra with center $K$ and invariant $-\lambda \in \mathbb{Q}/\mathbb{Z}$, and $h_i : = [D - \lambda_i : K]^{1/2}$. Noting that $G \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \simeq (GL_n)_{\mathbb{Q}_p}^{\text{Hom}_{\mathbb{Q}_p}(K, \mathbb{Q}_p)}$, consider a cocharacter $\mu : \mathbb{G}_m \to G$ over $\overline{\mathbb{Q}_p}$ represented (up to conjugacy) by

$$z \mapsto \prod_{\tau \in \text{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}_p})} (\text{diag}(z, \ldots, z, 1, \ldots, 1))_{p_{\tau}, q_{\tau}}$$

for nonnegative integers $(p_{\tau}, q_{\tau})_{\tau \in \text{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}_p})}$ such that $p_{\tau} + q_{\tau} = n$. Let $n' : = [K : \mathbb{Q}_p]n$. Given $\mu$ as above, set

$$(y_1, \ldots, y_{n'}) : = (1, \ldots, 1, 0, \ldots, 0).$$

For an element $b \in B(G)$ corresponding to $(r, \{\lambda_i\}_{1 \leq i \leq r}, \{m_i\}_{1 \leq i \leq r})$ as above, set

$$(x_1, \ldots, x_{n'}) : = (-\lambda_1, \ldots, -\lambda_i, \ldots, -\lambda_r).$$

Then $b \in B(G, \mu)$ if and only if

$$\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i \quad \text{for} \quad 1 \leq j < n' \quad \text{and} \quad \sum_{i=1}^{n'} x_i = \sum_{i=1}^{n'} y_i. \quad (9)$$

In particular the condition (9) implies that $1 \geq -\lambda_1 > \cdots > -\lambda_r \geq 0$.

**Definition 4.4.** By an isocrystal, we mean a pair $(V, \Phi)$ where $V$ is a finite-dimensional $L$-vector space and $\Phi : V \to V$ is a bijection such that $\Phi(lv) = \sigma(l)\Phi(v)$ for all $l, v \in V$. The **height** of $(V, \Phi)$ is the dimension of $V$ as an $L$-vector space. The height 1 isocrystal $(L, p^u\sigma)$ for $u \in \mathbb{Z}$ will be denoted $L(n)$. A morphism of isocrystals from $(V_1, \Phi_1)$ to $(V_2, \Phi_2)$ is a morphism $f : V_1 \to V_2$ of $L$-vector spaces such that $f \circ \Phi_1 = \Phi_2 \circ f$. We denote by Isoc the category of isocrystals.

**Definition 4.5.** By an isocrystal with $G$-structure or simply a $G$-isocrystal, we mean an exact faithful $\mathbb{Q}_p$-linear tensor functor

$$\mathfrak{M} : \text{Rep}_{\mathbb{Q}_p} G \to \text{Isoc}.$$ 

Here $\text{Rep}_{\mathbb{Q}_p} G$ denotes the category consisting of $(\rho, V)$ where $V$ is a finite dimensional $\mathbb{Q}_p$-vector space and $\rho : G \to GL(V)$ is a morphism of algebraic groups over $\mathbb{Q}_p$. Denote by $G$-Isoc the groupoid of isocrystals with $G$-structure. In other words, morphisms in $G$-Isoc are isomorphisms of $\mathbb{Q}_p$-linear tensor functors.

For each $\sigma$-conjugacy class $b \in B(G)$, choose a representative $\tilde{b} \in G(L)$. Define a functor $\mathfrak{M}_{\tilde{b}}$ by setting $\mathfrak{M}_{\tilde{b}}(\rho, V) = (V \otimes_{\mathbb{Q}_p} L, \rho_L(\tilde{b}) \cdot (1 \otimes \sigma))$. We will call $\mathfrak{M}_{\tilde{b}}$ an isocrystal with $G$-structure of type $b$. Note that if $\tilde{b}'$ is another representative of $b$, then $\mathfrak{M}_{\tilde{b}'}$ is canonically isomorphic to $\mathfrak{M}_{\tilde{b}}$.

**Lemma 4.6.** The association $b \mapsto \mathfrak{M}_{\tilde{b}}$ defines a natural bijection from $B(G)$ onto the set of isomorphism classes of isocrystals with $G$-structure.
\textbf{Proof.} The inverse map to \( b \mapsto \mathcal{M}_b \) is given in \cite[Rem 3.5]{RR96}, using Steinberg’s theorem on vanishing of \( H^1 \)-cohomology. \( \square \)

\textbf{Lemma 4.7.} There is a natural isomorphism \( \text{Aut}_{G-\text{Isoc}}(\mathcal{M}_b) \cong J_b(\mathbb{Q}_p) \).

\textbf{Proof.} Define \( \mathcal{M}^{fib}_b : \text{Rep}_\mathbb{Q}_p G \to \text{Vect}_L \) to be the composition of \( \mathcal{M}_b \) with the forgetful functor where \( \text{Vect}_L \) denotes the category of \( L \)-vector spaces. Then the automorphisms of the functor \( \mathcal{M}_b \) are those automorphisms of \( \mathcal{M}^{fib}_b \) preserving isocrystal structures. Namely,

\[ \text{Aut}(\mathcal{M}_b) = \{ \varphi \in \text{Aut}(\mathcal{M}^{fib}_b) : \varphi(\rho, V)\rho_L(\tilde{b})(1 \otimes \sigma) = \rho_L(\tilde{b})(1 \otimes \sigma)\varphi(\rho, V) \text{ on } V_L \text{ for all } (\rho, V) \}. \]

Using the isomorphism \( \text{Aut}(\mathcal{M}^{fib}_b) \cong \text{G}(L) \), we have

\[ \text{Aut}(\mathcal{M}_b) = \{ g \in \text{G}(L) : g\rho_L(g_L(\tilde{b}))(1 \otimes \sigma) = \rho_L(\tilde{b})(1 \otimes \sigma)g \text{ for all } (\rho, V) \} = \{ g \in \text{G}(L) : g\sigma = \tilde{\sigma}g \} = J_b(\mathbb{Q}_p) \]

\( \square \)

\textbf{Remark 4.8.} Define a category \( \mathcal{B}(G) \) whose objects are elements of \( \text{G}(L) \) and \( \text{Mor}(\tilde{b}_1, \tilde{b}_2) := \{ g \in \text{G}(L) : g\tilde{b}_1\sigma = \tilde{b}_2\sigma g \} \). Lemma 4.6 and Lemma 4.7 mean that \( \mathcal{B}(G) \) is equivalent to \( G-\text{Isoc} \) via \( b \mapsto \mathcal{M}_b \).

\textbf{Example 4.9.} Consider \((B, *, V, \langle \cdot, \cdot \rangle)\) and the associated \( \mathbb{Q}_p \)-group \( G \) as below.

- \( B \) is a finite dimensional semisimple algebra over \( \mathbb{Q}_p \) with involution \( * \) such that \( F := Z(B) \) is a product of unramified extensions over \( \mathbb{Q}_p \).
- \( V \) is a finite \( B \)-module with a \(*\)-Hermitian pairing \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}_p \).
- \( G \) is the \( \mathbb{Q}_p \)-group such that for any \( \mathbb{Q}_p \)-algebra \( R \),

\[ G(R) = \{ g \in \text{End}_B(V) \otimes_{\mathbb{Q}_p} R \mid \langle gv, gw \rangle = \varpi(g)\langle v, w \rangle \text{ for some } \varpi(g) \in R^\times, \forall v, w \in V \}. \]

For instance, the datum as above is obtained by taking \( \otimes_{\mathbb{Q}_p} \) of a Shimura datum (see §5). Define a category \( G-\text{Isoc}' \) whose objects are tuples \((V', \Phi', C', \langle \cdot, \cdot \rangle', \i')\) where

- \((V', \Phi')\) is an isocrystal such that \( V' \simeq V_L \) as \( L \)-vector spaces.
- \( C' \) is a height 1 isocrystal.
- \( \i' : B \to \text{End}_{\text{Isoc}}(V', \Phi') \) is a \( \mathbb{Q}_p \)-algebra map.
- \( \langle \cdot, \cdot \rangle' : V' \otimes V' \to C' \) is a map of isocrystals such that the underlying map on \( L \)-vector spaces defines an \( L \)-linear nondegenerate and alternating \(*\)-Hermitian pairing (with respect to \( B \)-action).
- A morphism from \((V_1', \Phi_1', C_1', \langle \cdot, \cdot \rangle_1', \i_1')\) to \((V_2', \Phi_2', C_2', \langle \cdot, \cdot \rangle_2', \i_2')\) is a pair of isomorphisms of isocrystals \( \alpha : V_1' \to V_2' \) and \( \beta : C_1' \to C_2' \) such that \( \langle \cdot, \cdot \rangle_2 \circ (\alpha, \alpha) = \beta \circ \langle \cdot, \cdot \rangle_1 \).

Denote by \( \rho \) the standard representation \( G \hookrightarrow GL(V) \), which is defined over \( \mathbb{Q}_p \). It can be shown that \( \mathfrak{M} \mapsto \mathfrak{M}(\rho) \) gives an equivalence over \( \mathbb{Q}_p \). It can be shown that \( \mathfrak{M} \mapsto \mathfrak{M}(\rho) \) gives an equivalence of categories \( G-\text{Isoc} \simeq G-\text{Isoc}' \). (cf. \cite[Rem 3.4.(v)]{RR96}) If \( b \) belongs to \( B(G, -\mu) \) where \( G \) and \( \mu \) arise from a Shimura datum (§5), then we may take \( C' = L(-1) \) for isocrystals of type \( b \).

We will use the terminology of Barsotti-Tate groups (or simply BT-groups) following \cite[Ch1]{Mes72}.
Definition 4.10. Let $\Sigma_1$ and $\Sigma_2$ be BT-groups over $S$. A morphism $f : \Sigma_1 \to \Sigma_2$ is called an isogeny if $f$ is an epimorphism and $\ker f$ is a finite locally free group scheme over $S$. A quasi-isogeny from $\Sigma_1$ to $\Sigma_2$ is a global section $f$ of the sheaf $\text{Hom}_S(\Sigma_1, \Sigma_2) \otimes \mathbb{Q}$ such that any point of $S$ has a Zariski neighborhood where $p^n f$ is an isogeny for some positive integer $n$.

Definition 4.11. A polarization (resp. quasi-polarization) of a BT-group $\Sigma$ over $S$ is an isogeny (resp. a quasi-isogeny) $\lambda : \Sigma \to \Sigma'$ over $S$ such that $\lambda^\vee = [-1] \lambda$ (via the canonical isomorphism $\Sigma \simeq \Sigma^\vee\vee$).

Denote by $\text{BT}_S^0$ the category whose objects are BT-groups over $S$ and morphisms are given by $\text{Hom}_S(\Sigma_1, \Sigma_2) \otimes \mathbb{Q}$. An endomorphism algebra in $\text{BT}_S^0$, written as $\text{End}_S^0(\Sigma)$, is a $\mathbb{Q}_p$-algebra.

Consider the case $S = \text{Spec } \mathbb{F}_p$. Let $D$ be the contravariant Dieudonné functor from the category of BT-groups over $\mathbb{F}_p$ to the category of finite free $W$-modules equipped with $F$ and $V$ actions which are semilinear for $\sigma$ and $\sigma^{-1}$, respectively, such that $F V = V F = p$. This is an anti-equivalence of categories. (For instance, see [Dem72].) In this paper, we will use a covariant version of the Dieudonné functor by taking dual vector spaces. The functor $\mathcal{V}$ sends a BT-group $\Sigma$ over $\mathbb{F}_p$ to the isocrystal

$$\mathcal{V}(\Sigma) := (\text{Hom}_\mathbb{L}(D(\Sigma) \otimes \mathbb{W} L, L), F^*)$$

where $F^*$ is induced by the $F$ action on $D(\Sigma)$. We see that $\mathcal{V}$ is a fully faithful functor from $\text{BT}_S^0$ to $\text{Isoc}$. Observe that $\mathcal{V}(\mu_{p^n}) = L(-1)$ as an isocrystal and that more generally $\mathcal{V}(\Sigma)$ has the set of slopes $\{ -\lambda_i \}$ if $\Sigma$ has $\{ \lambda_i \}$. The usual height of a BT-group $\Sigma$ is equal to the height of the isocrystal $\mathcal{V}(\Sigma)$.

Remark 4.12. The categories $\text{Isoc}$ and $\text{BT}_S^0$ are $\mathbb{Q}_p$-linear categories. In particular, a morphism need not have an inverse morphism. However when it comes to isocrystals or BT-groups with $G$-structure, we will restrict our attention to invertible morphisms in the categories.

Now we consider BT-groups with PEL structure. Recall the notation $B, F, V, G$ from Example 4.9. Fix a maximal order $\mathcal{O}_B$ of $B$.

Definition 4.13. The category $\text{BT}^{0,G}_S$ has as objects the triples $(\Sigma, \lambda, i)$ such that

- $\Sigma$ is a BT-group over $S$,
- $\lambda : \Sigma \to \Sigma'$ is a quasi-polarization, and
- $i : \mathcal{O}_B \to \text{End}_S(\Sigma)$ is a $\mathbb{Z}_p$-algebra morphism such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for all $b \in \mathcal{O}_B$.

The morphisms from $(\Sigma_1, \lambda_1, i_1)$ to $(\Sigma_2, \lambda_2, i_2)$ are the quasi-isogenies $f : \Sigma_1 \sim \Sigma_2$ satisfying two conditions: $f \circ i_1(b) = i_2(b) \circ f$ for all $b \in \mathcal{O}_B$ and $\lambda_1 = \gamma f \circ \lambda_2 \circ f'$ for some $\gamma \in \mathbb{Q}_p$. The automorphism group of $(\Sigma, \lambda, i)$ is denoted by $\text{Aut}^G(\Sigma, \lambda, i)$.

The previous functor $\mathcal{V}$ can be made to incorporate the $G$-structure in this case. Given $(\Sigma, \lambda, i)$, we have the isocrystal $(V', \Phi') := \mathcal{V}(\Sigma)$. By functoriality, the map $i$ induces a $\mathbb{Q}_p$-algebra map $i' : B \to \text{End}_{\text{Isoc}}(V', \Phi')$. The map of BT-groups $\Sigma \times \Sigma \to \mu_{p^n}$ coming from $\lambda$ induces a map of isocrystals $(\cdot, \cdot)' : V' \otimes V' \to L(-1)$. So $\mathcal{V}$ can be extended to $\text{BT}^{0,G}_\mathbb{F}_p$ as follows, using the notation of Example 4.9. The extended functor is again fully faithful.

$$\mathcal{V} : \text{BT}^{0,G}_\mathbb{F}_p \to G\text{-Isoc}'$$

$$(\Sigma, \lambda, i) \mapsto (V', \Phi', L(-1), (\cdot, \cdot)', i').$$

From Lemma 4.7 we deduce the following.
**Lemma 4.14.** If \( \mathcal{V}(\Sigma, \lambda, i) \) is a \( G \)-isocrystal of type \( b \) then there is an isomorphism \( J_b(\mathbb{Q}_p) \cong \text{Aut}^0(\Sigma, \lambda, i) \) which is canonical up to an inner automorphism.

**Remark 4.15.** Using the \( B \)-action and the pairing on \( V \), the isocrystal \( (V \otimes_{\mathbb{Q}_p} L, \tilde{b}(1 \otimes \sigma)) \) naturally extends to an object of \( G\text{-Isoc}' \), which is canonically isomorphic to the image of \( \mathfrak{M}_b \) under \( G\text{-Isoc} \cong G\text{-Isoc}' \). Call this object \( \mathcal{V}(\tilde{b}) \). The condition that \( \mathcal{V}(\Sigma, \lambda, i) \) is a \( G \)-isocrystal of type \( b \) is equivalent to the condition that \( \mathcal{V}(\Sigma, \lambda, i) \cong \mathcal{V}(\tilde{b}) \) in \( G\text{-Isoc}' \).

## 5 PEL-type Shimura varieties and Igusa varieties

We fix the choice of the rational primes \( p \) and \( l \). We always assume that \( p \neq l \).

**Definition 5.1.** A partial Shimura PEL datum is a quadruple \( (B, *, V, \langle \cdot, \cdot \rangle) \) where

- \( B \) is a finite-dimensional simple \( \mathbb{Q} \)-algebra.
- \( * \) is an involution of \( B \). We assume \( * \) is positive, i.e. \( \text{tr}(bb^*) > 0 \) for every \( b \in B^* \).
- \( V \) is a finite semisimple \( B \)-module.
- \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q} \) is a \(*\)-Hermitian pairing with respect to \(*\)-action.

We will denote the center of \( B \) by \( F \). We associate a \( \mathbb{Q} \)-group \( G \) to \( (B, *, V, \langle \cdot, \cdot \rangle) \) by \( G(R) = \{ g \in \text{End}_{B\otimes R}(V \otimes R) \mid \exists \varpi(g) \in R^*, \langle gv_1, gv_2 \rangle = \varpi(g)\langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in V \otimes R \} \) for any \( \mathbb{Q} \)-algebra \( R \).

Given \( (B, *, V, \langle \cdot, \cdot \rangle) \) as above, we can define a simple algebra \( C := \text{End}_{B}(V) \) with an involution \( \# \). The involution \( \# \) is uniquely determined by the following relation: for each \( c \in C \), \( \langle cv, w \rangle = \langle v, c^\# w \rangle \) for all \( v, w \in V \).

**Definition 5.2.** An (unramified) integral Shimura PEL datum is a septuple \( (B, C_B, *, V, \Lambda_0, \langle \cdot, \cdot \rangle, h) \) where

- \( (B, *, V, \langle \cdot, \cdot \rangle) \) is a partial Shimura PEL datum where \( B \otimes \mathbb{Q} \mathbb{Q}_p \) is isomorphic to a product of matrix algebras over unramified extension fields of \( \mathbb{Q}_p \).
- \( h : C \to C_\mathbb{R} \) is an \( \mathbb{R} \)-algebra homomorphism with involution (i.e. \( \forall z \in C, h(z^*) = h(z)^* \)) such that the bilinear pairing \( \langle v, w \rangle \mapsto \langle v, h(\sqrt{-1})w \rangle \) is symmetric and positive definite.
- \( C_B \) is a \( \mathbb{Z}_{(p)} \)-maximal order in \( B \) that is preserved by \(*\) such that \( \mathcal{O}_B \otimes \mathbb{Z} \mathbb{Z}_p \) is a maximal order in \( B_{\mathbb{Q}_p} \).
- \( \Lambda_0 \) is a \( \mathbb{Z}_p \)-lattice in \( V_{\mathbb{Q}_p} \) that is preserved by \( \mathcal{O}_B \) and self-dual for \( \langle \cdot, \cdot \rangle \).

Put \( F := Z(B) \) and \( F^+ := F^{* = 1} \). The above definition implies that for an integral Shimura PEL datum, \( F \) is a finite extension of \( \mathbb{Q} \) unramified at \( p \) and \( G \) is unramified over \( \mathbb{Q}_p \). Let us define a \( \mathbb{Q} \)-group \( G_0 \) by the exact sequence \( 1 \to G_0 \to G \to \mathbb{G}_m \to 1 \). Let \( n := [B : F]^{1/2} \) and \( d := [F^+ : \mathbb{Q}] \).

The map \( h \) in the datum induces a group homomorphism \( \text{Res}_{C/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}} \), again written as \( h \) by abuse of notation. Define a map \( \mu_h : \mathbb{G}_m \to G \) defined over \( C \) by the composition

\[
C^x \hookrightarrow C^x \times C^x \simeq (C \otimes \mathbb{C})^x \xrightarrow{(h, \text{id})} (C \otimes \mathbb{R} \mathbb{C})^x.
\]
The first arrow is the embedding \( z \mapsto (z, 1) \) and the inverse of the second map is induced by the map \( z_1 \oplus z_2 \mapsto (z_1 z_2, z_1 \bar{z}_2) \) on the underlying \( \mathbb{C} \)-algebras. We often write \( \mu \) for \( \mu_t \) when there is no confusion. We obtain a decomposition \( V_\mathbb{C} = V^0 \oplus V^1 \) where \( V^0 \) (resp. \( V^1 \)) is the \( \mathbb{C} \)-vector space on which \( \mu(z) \) acts by 1 (resp. \( z \)). On the other hand, as \( B_\mathbb{C} \simeq \prod_{\tau \in \text{Hom}_\mathbb{Q}(F, \mathbb{C})} M_n(\mathbb{C}) \), we have a corresponding decomposition \( V_\mathbb{C} = \oplus_\tau V_\tau \). The two decompositions are compatible in the sense that we have further decomposition \( V_\tau = V^0_\tau \oplus V^1_\tau \) as \( B \otimes_{\mathbb{C}, \tau} \mathbb{C} \)-modules such that \( V^0 = \oplus_\tau V^0_\tau \) and \( V^1 = \oplus_\tau V^1_\tau \) for each \( \tau \in \text{Hom}_\mathbb{Q}(F, \mathbb{C}) \). We define an integer \( p_\tau := (\dim_\mathbb{C} V^1_\tau)/n \) for each \( \tau \). The field of definition \( E \) for the \( B_\mathbb{C} \)-module \( V^1 \) is called the reflex field. Note that \( E \) is naturally a subfield of \( \mathbb{C} \). As \( F \) is unramified at \( p \), so is the reflex field \( E \).

An (integral) Shimura PEL datum falls into types (A), (C) or (D). Note that the existence of the quadruples on the right consists of

- \( \mathcal{A} \) is an abelian scheme over \( S \).
- \( \lambda : \mathcal{A} \to \mathcal{A}^\vee \) is a prime-to-\( p \) polarization.
- \( i : \mathcal{O}_B \to \text{End}(\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \) such that \( \lambda \circ i(b) = i(b^*)^\vee \circ \lambda \), \( \forall b \in \mathcal{O}_B \).
- \( \bar{\eta}^p \) is a \( \pi_1(S, s) \)-invariant \( U^p \)-orbit of isomorphisms of \( B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty-p} \)-modules \( \eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty-p} \simeq V^p A \), which take the pairing \( \langle \cdot, \cdot \rangle \) to the \( \lambda \)-Weil pairing up to \( (\mathbb{A}^{\infty-p})^* \)-multiples. Here \( s \) is any geometric point of \( S \). (For any two geometric points \( s \) and \( s' \), \( \bar{\eta}^p \) may be canonically identified.)
- (Determinant condition) An equality of polynomials \( \det_{\mathcal{O}_A}(b \mid \text{Lie} A) = \det_F(b \mid V^1) \) holds for all \( b \in \mathcal{O}_B \), in the sense of \([\text{Kot92, §5}]\).
- Two quadruples \( (A_1, \lambda_1, i_1, \bar{\eta}_1^p) \) and \( (A_2, \lambda_2, i_2, \bar{\eta}_2^p) \) are equivalent if there is a prime-to-\( p \) isogeny \( A_1 \to A_2 \) taking \( \lambda_1, i_1, \bar{\eta}_1^p \) to \( \gamma \lambda_2, i_2, \bar{\eta}^p_2 \) for some \( \gamma \in \mathbb{Z}_p^\times \).

If \( U^p \) is sufficiently small, this functor is representable by a quasi-projective smooth scheme over \( \mathcal{O}_{K,(p)} \) of finite type, which we call \( X_U \). For the proof of representability, see the comment in \([\text{Kot92, p.391}]\). Henceforth, we will write \( (\mathcal{A}, \lambda^{\text{univ}}, i^{\text{univ}}, \bar{\eta}^p)^{\text{univ}} \) for the universal object. (So \( \mathcal{A} \) is an abelian scheme over \( X_U \).)

Before considering the special fiber of \( X_U \), fix an isomorphism \( \iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C} \), which determines an embedding \( E \hookrightarrow \overline{\mathbb{Q}}_p \) and thus a place \( w \) of \( E \) over \( p \). We also fix a reduction map \( \iota_p : \mathcal{O}_{\overline{Q}^r} \to \overline{\mathbb{F}}_p \).
Observe that the maps $E \hookrightarrow \overline{\mathbb{Q}}_p$ and $\iota_p$ pin down the composite map $\mathcal{O}_E \hookrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_p} \to \mathbb{F}_p$. We choose an embedding $k(w) \hookrightarrow \mathbb{F}_p$ so that the reduction map $\mathcal{O}_E \to k(w)$ followed by $k(w) \hookrightarrow \mathbb{F}_p$ coincides with the last composite map.

Using $\iota_p$, we may view $\mu = \mu_b$ as a map $\mathbb{G}_m \to G$ defined over $\overline{\mathbb{Q}}_p$. Define $\mu_1 \in X^*(Z(\overline{G})^\Gamma(p))$ as follows. Choosing a maximal torus $\hat{T}$ of $\hat{G}$, obtain a Weyl group orbit of $\hat{\mu}$ in $X^*(\hat{T})$. Then $\mu_1$ is the restriction of $\hat{\mu}$ to $Z(\overline{G})^\Gamma(p)$. It is easily seen that $\mu_1$ is independent of the choice of $h$ (in its $G(\mathbb{R})$-conjugacy class), $\hat{T}$ and $\hat{\mu}$. Clearly this definition of $\mu_1$ is compatible with the one in the last section.

Put $\overline{X}_U := X_U \times_{\mathcal{O}_{E,(p)}} k(w)$. Consider the Newton polygon stratification

$$\overline{X}_U = \coprod_{b \in B(G)} \overline{X}_U^{(b)}$$

where each stratum is set-theoretically given by

$$\overline{X}_U^{(b)} := \{ x \in \overline{X}_U : (\mathscr{A}_b[p^\infty], \lambda_x^{\text{univ}}, i_x^{\text{univ}}) \simeq (\Sigma, \lambda_\Sigma, i_\Sigma) \text{ in } BT_{\mathbb{F}_p}^0 \}. $$

Note that any $(\mathscr{A}_b[p^\infty], \lambda_x^{\text{univ}}, i_x^{\text{univ}})$ is a BT-group of type $b$ for some $b \in B(G, -\mu)$ by the moduli problem. Thus $\overline{X}_U^{(b)} = \emptyset$ if $b \notin B(G, -\mu)$. As $\overline{X}_U^{(b)}$ is a locally closed subset of $\overline{X}_U$, we give $\overline{X}_U^{(b)}$ the reduced subscheme structure.

From now on, we fix $b \in B(G, -\mu)$ and focus on a single stratum $\overline{X}_U^{(b)}$. Fix once and for all a decent representative $\bar{b} \in G(L)$ of $b$ (Definition 4.1). We will keep writing $\overline{X}_U^{(b)}$ and $\mathscr{A}$ for $\overline{X}_U^{(b)} \times_{k(w)} \mathbb{F}_p$ and $\mathscr{A} \times_{k(w)} \mathbb{F}_p$ by abuse of notation.

The result of [Win05, Thm 2] ensures the existence of a BT-group $\Sigma = \Sigma_b$ over $\mathbb{F}_p$ equipped with a polarization $\lambda_\Sigma : \Sigma \to \Sigma'$ and a $\mathbb{Z}_p$-algebra map $i_\Sigma : \mathcal{O}_E \otimes \mathbb{Z}_p \to \text{End}_{\mathbb{F}_p}(\Sigma)$ such that

(i) $\forall (\Sigma, \lambda_\Sigma, i_\Sigma)$ is a $G$-isocrystal of type $b$. (See §4.)

(ii) $\Sigma = \oplus_{i=1}^r \Sigma^i$ where $\Sigma^i$ has slope $\lambda_i$, and $1 \geq \lambda_1 > \cdots > \lambda_r \geq 0$.

(iii) (Determinant condition) An equality of polynomials $\det_{\mathfrak{g}}(a | \text{Lie } \Sigma) = \det_{\mathfrak{g}}(a | V^1)$ holds for all $a \in \mathcal{O}_E$, in the sense of [Kot92, §5]. (The polynomial on the left (resp. right) hand side has coefficients in $\mathbb{F}_p$ (resp. $\mathcal{O}_{E,(p)}$)). The two are compared via $\iota_p : \mathcal{O}_{E,(p)} \to \mathbb{F}_p$.

(iv) The degree of $\lambda_\Sigma$ is prime to $p$.

We fix a choice of such $(\Sigma, \lambda_\Sigma, i_\Sigma)$. Define the $\mathbb{F}_p$-subscheme $C_{b,\mathbb{F}_p}$ of $\overline{X}_U^{(b)}$ as follows. Set-theoretically

$$C_{b,\mathbb{F}_p} := \{ x \in \overline{X}_U : (\mathscr{A}_b[p^\infty], \lambda_x^{\text{univ}}, i_x^{\text{univ}}) \simeq (\Sigma, \lambda_\Sigma, i_\Sigma) \text{ in } BT_{\mathbb{F}_p}^0 \}. $$

We give $C_{b,\mathbb{F}_p}$ the reduced closed subscheme structure, which makes sense since $C_{b,\mathbb{F}_p}$ is Zariski closed in $\overline{X}_U^{(b)}$. Recall that $\overline{X}_U^{(b)}$ is smooth over $\mathbb{F}_p$. In fact, $C_{b,\mathbb{F}_p}$ is also smooth over $\mathbb{F}_p$ ([Man05, prop 1]). Note that $C_{b,\mathbb{F}_p}$ could be an empty set without the condition (iii) on $(\Sigma, i_\Sigma, \lambda_\Sigma)$.

As we have a natural immersion of $C_{b,\mathbb{F}_p}$ into $\overline{X}_U = \overline{X}_{U \times \mathbb{F}_p}$, we may pull back the BT-group (with additional structure) of the universal abelian scheme $\mathscr{A}$ to define a BT-group $\mathscr{A}$ over $C_{b,\mathbb{F}_p}$ with additional structure.
Then \( \mathcal{G} \) is completely slope divisible ([Man04, 3.2.3]), which means that \( \mathcal{G} \) has a slope filtration (0) = \( \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_r = \mathcal{G} \) such that each quotient \( gr^i \mathcal{G} := \mathcal{G}_i / \mathcal{G}_{i-1} \) has a pure slope \( \lambda_i \) and that \( \mathcal{G}_i \) is slope divisible with respect to \( \lambda_i \) where 1 \( \geq \lambda_1 > \cdots > \lambda_r \geq 0 \). Of course, the numbers \( r \) and \( \lambda_i \) are the same as the ones for \( \Sigma \) in the above. The subquotients \( gr^i \mathcal{G} \) inherit the additional structure \( i \) and \( \lambda \) from \( \mathcal{G} \). For more detail, see [Man05, §3].

Now we are ready to define Igusa varieties. They depend on our choice of \((\Sigma, i_{\Sigma}, \lambda_{\Sigma})\) as \( C_{b,U^p} \) does.

**Definition 5.3.** Let \( m \) be a positive integer. The Igusa variety \( Ig_{b,U^p,m} \) is defined to be the moduli space of the set of the following isomorphisms of finite flat group schemes over \( C_{b,U^p} \)

\[
j_{m,i}^{uni} : \Sigma[p^m] \times_{p} C_{b,U^p} \xrightarrow{\sim} gr^{i} \mathcal{G}[p^m], \quad 1 \leq i \leq r\]

where \( j_{m,i}^{uni} \) extends étale locally to all higher levels \( m' \geq m \) and preserves \( \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \)-actions and polarizations, the latter up to \((\mathbb{Z}/p^m\mathbb{Z})^\times\)-multiples.

The moduli problem for Igusa varieties is proved to be representable. (See the remark following Definition 3 of [Man05].) Note that there is a natural projection map from \( Ig_{b,U^p,m} \) to \( C_{b,U^p} \) forgetting the data \( j_{m,i}^{uni} \). This map \( Ig_{b,U^p,m} \to C_{b,U^p} \) is finite étale and Galois ([Man05, prop.5]). Thus \( Ig_{b,U^p,m} \) is smooth over \( \overline{\mathbb{F}}_p \). However \( Ig_{b,U^p,m} \) is usually not proper over \( \overline{\mathbb{F}}_p \).

Choose an irreducible algebraic representation \( \xi \) of \( G \) on a finite dimensional \( \overline{Q}_l \)-vector space. It naturally defines a lisse \( \overline{Q}_l \)-sheaf \( \mathcal{L}_\xi \) on \( X_{U^p \times U^{p^m}} \) whenever \( U^p \) is small enough. (See [Kot92, §6] for instance.) The pullback of \( \mathcal{L}_\xi \) to \( Ig_{b,U^p,m} \) is again denoted \( \mathcal{L}_\xi \) by abuse of notation. Let \( Ig_b \) denote the projective system \( \lim_{\longleftarrow} Ig_{b,U^p,m} \) and define

\[
H^k(Ig_b, \mathcal{L}_\xi) := \lim_{U^p,m} H^k(Ig_{b,U^p,m}, \mathcal{L}_\xi), \quad Ig_b(\overline{\mathbb{F}}_p) := \lim_{U^p,m} Ig_{b,U^p,m}(\overline{\mathbb{F}}_p).
\]

We describe the action of \( G(k^{\infty,p}) \) on the projective system \( Ig_b \). In terms of the moduli data, \( g \in G(k^{\infty,p}) \) acts as

\[
(A, \lambda, i, \eta^p, \{j_{m,i}\}) \mapsto (A, \lambda, i, \eta^p \circ g, \{j_{m,i}\}).
\]

We remark that \( g \) maps \( Ig_{b,U^p,m} \) to \( Ig_{b,g^{-1}U^p,g,m} \).

Defining the action of \( J_b(Q_p) \) is more subtle. Recall from Lemma 4.14 that

\[
J_b(Q_p) \simeq \text{Aut}^0(\Sigma, \lambda_{\Sigma}, i_{\Sigma}).
\]

We fix this isomorphism and define another group consisting of genuine automorphisms (not quasi-isogenies) in the group \( J_b(Q_p) \):

\[
\Gamma_b := \text{Aut}(\Sigma) \cap J_b(Q_p).
\]

Choose a positive integer \( s \) such that \( s\lambda_i \in \mathbb{Z} \) for all \( i \). Define an element in \( \text{End}^0(\Sigma) \), formally written as \( fr^{-s} \), which acts as \( p^{-s}\lambda_i \) on \( \Sigma^i \) for each \( i \). Observe that \( fr^{-s} \) belongs to the center of \( J_b(Q_p) \). Denote by \( fr^s \in J_b(Q_p) \) the inverse of \( fr^{-s} \) in \( J_b(Q_p) \).

We recall from [Man05, §4] the definition of a submonoid \( S_b \) of \( J_b(Q_p) \). For \( \delta \in J_b(Q_p) \), suppose that \( \delta^{-1} \) is an isogeny. Any \( \delta \) may be written as \( \delta = (\delta_i)_{i=1}^r \) with \( \delta_i \in \text{End}^0(\Sigma^i) \). For each \( i \in [1, r] \), we define \( e(\delta_i) \) and \( f(\delta_i) \) to be the minimal and maximal integers such that \( \ker[p^f(\delta_i)] \subset \ker[\delta^{-1}] \subset \ker[p^{e(\delta_i)}] \).

The monoid \( S_b \) is defined by

\[
S_b := \{ \delta \in J_b(Q_p) \mid \delta^{-1} \text{ is an isogeny, } f(\delta_{i-1}) \geq e(\delta_i), \ \forall 2 \leq i \leq r \}
\]
We list some properties of $S_b$. First, the relation $\Gamma_b \subset S_b \subset J_b(\mathbb{Q}_p)$ holds. Second, $S_b$ contains $p^{-1}$ and $fr^{-s}$. Finally, $J_b$ is generated by $S_b$ and the two elements $p$, $fr^s$ as a monoid.

An element $\gamma \in \Gamma_b$ acts on $I_{g_b,U^p,m}$ as

$$(A, \lambda, i, \eta, \{j_m,i\}) \mapsto (A, \lambda, i, \eta^\gamma, \{j_m,i \circ \gamma\})$$

and this action extends to the projective system $I_{g_b}$. It is possible to extend this to an action of $S_b$ on $I_{g_b}$ (see Lemma 5 and the paragraph below it in [Man05]), but not to an action of $J_b(\mathbb{Q}_p)$. Nevertheless, the $S_b$-action on the cohomology space $H^k_c(I_{g_b}, \mathcal{L}_\xi)$ does extend to a $J_b(\mathbb{Q}_p)$-action since the actions of $p^{-1}$, $fr^{-s} \in S_b$ on $H^k_c(I_{g_b}, \mathcal{L}_\xi)$ are invertible ([Man05, Lem 6]). We define

$$U_p(m) := \ker(\Gamma_b \to \text{Aut}(\Sigma[p^m], \lambda_\Sigma, i_\Sigma))$$

which are subgroups of $\Gamma_b$. They form an open basis around the identity in the group $\Gamma_b$. We know from ([Man05, Prop 4, Prop 7]) that

(i) $H^k_c(I_{g_b,U^p,m}, \mathcal{L}_\xi) \cong H^k_c(I_{g_b}, \mathcal{L}_\xi)^{U_p \times U_p(m)}$,

(ii) The natural map $I_{g_b,U^p,m} \to C_{b,U^p}$ is finite and Galois with Galois group $\Gamma_b/U_p(m)$.

In particular, the action of $G(A^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $H^k_c(I_{g_b}, \mathcal{L}_\xi)$ is continuous and admissible. Define

$$H_c(I_{g_b}, \mathcal{L}_\xi) := \sum_k (-1)^k H^k_c(I_{g_b}, \mathcal{L}_\xi),$$

as an object of Groth($G(A^{\infty,p}) \times J_b(\mathbb{Q}_p)$). Our primary goal is to study this space via a counting point formula.

6 From trace to counting points

Denote by $\text{char}_H$ the function which has the value 1 on $H$ and 0 outside $H$. Set $U^p(m) := U^p \times U_p(m)$ for any $m \in \mathbb{Z}_{>0}$. Any function $\varphi \in C_c^\infty(G(A^{\infty,p}) \times J_b(\mathbb{Q}_p))$ can be written as

$$\varphi = \sum_{g \in I} \alpha_g \text{char}_{U^p(m)gU^p(m)}$$

for some $\alpha_g \in \mathbb{C}$, $g \in I$ where $I$ is a finite subset of $G(A^{\infty,p}) \times J_b(\mathbb{Q}_p)$. So the computation of $\text{tr}(\varphi|H_c(I_{g_b}, \mathcal{L}_\xi))$ comes down to the case where $\varphi$ is of the form $\text{char}_{U^p(m)gU^p(m)}$.

Write $U^p(m)gU^p(m) = \bigsqcup gU^p(m)$, which is a finite union. Then the following double coset action is well-defined.

$$\text{tr}([U^p(m)gU^p(m)]H_c(I_{g_b,U^p,m}, \mathcal{L}_\xi)) := \sum_k \sum_{i \geq 0} (-1)^k \text{tr}(g_i|H^k_c(I_{g_b,U^p,m}, \mathcal{L}_\xi))$$

(10)

It is an elementary matter to check that

$$\text{tr}(\text{char}_{U^p(m)gU^p(m)}H_c(I_{g_b}, \mathcal{L}_\xi)) = \text{vol}(U^p(m))\text{tr}([U^p(m)gU^p(m)]H_c(I_{g_b,U^p,m}, \mathcal{L}_\xi)).$$

(11)

We recall the notion of fixed points of an algebraic correspondence in general. Let $\alpha, \beta : Y \to X$ be morphisms of $k$-varieties where $k$ is an algebraically closed field. The correspondence induced by
The maps \( pr_1 \) and \( pr_2 \) are the projections onto the first and the second components. 

Then we have an induced map \( Y(k) \to X(k) \times X(k) \). Define the set of fixed points

\[
\mathrm{Fix}(\gamma) := \{ y \in Y(k) | \alpha(y) = \beta(y) \}.
\]  

Now we consider correspondences on Igusa varieties. We would like to interpret the action of \([U^p(m)gU^p(m)]\) in (10) as an algebro-geometric correspondence to which Fujiwara’s trace formula can be applied. For this interpretation, we need to assume that \( g \in G(\mathbb{A}^\infty_p) \times \mathcal{S}_b \). Then we may choose a small enough subgroup \( V^p(m') \) contained in \( U^p(m) \cap gU^p(m)g^{-1} \) so that the map given by \( g \) below is well-defined ([Man05, Lem 6]). The correspondence \([U^p(m)gU^p(m)]\) is understood as in the following diagram where \( pr \) means the natural projection.

In practice, we may often regard \([U^p(m)gU^p(m)]\) as a set-theoretic correspondence. Recall that \( \mathrm{Ig}_b(F_p) = \varprojlim_{U^p,m} \mathrm{Ig}_{b,U^p,m}(F_p) \) and \( \mathrm{Ig}_{b,U^p,m}(F_p) = \mathrm{Ig}_b(F_p)/U^p(m) \) as sets with right \( G(\mathbb{A}^\infty_p) \times J_b(Q_p) \)-action. On the level of \( F_p \)-points, (13) fits into the following diagram. Note that in general the map \( g : \mathrm{Ig}_b(F_p)/(U^p(m) \cap gU^p(m)g^{-1}) \to \mathrm{Ig}_b(F_p)/U^p(m) \) does not come from a map of algebraic varieties.

When we consider \([U^p(m)gU^p(m)]\) as an algebraic correspondence on \( \mathrm{Ig}_{b,U^p,m} \), we need to think of it a priori as (13). When dealing with \( F_p \)-points, \([U^p(m)gU^p(m)]\) may also be understood as (14). The set of fixed points under \([U^p(m)gU^p(m)]\) as a set-theoretic correspondence will be understood as

\[
\mathrm{Fix}([U^p(m)gU^p(m)]) = \{ x \in \mathrm{Ig}_b(F_p)/(U^p(m) \cap gU^p(m)g^{-1}) | x = xg \text{ in } \mathrm{Ig}_b(F_p)/U^p(m) \}.
\]

The virtue of the algebro-geometric interpretation is that we may apply Fujiwara’s trace formula to compute the trace. (See the proof of Lemma 6.3.) The formula that we would like to have is

\[
\text{tr}([U^p(m)gU^p(m)])\mathcal{H}_*(\mathrm{Ig}_{b,U^p,m}, \xi) = \sum_{x \in \mathrm{Fix}([U^p(m)gU^p(m)])} \text{tr}([U^p(m)gU^p(m)](\mathcal{L}_x)).
\]

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where $\text{Fix}([U^p(m)gU^p(m)])$ is the set of fixed points in the sense of (12) under the algebro-geometric correspondence $[U^p(m)gU^p(m)]$ understood as (13) (for a chosen subgroup $V^p(m')$ there). But once we know the validity of (16), it is an easy exercise to check that the same identity still holds if $[U^p(m)gU^p(m)]$ is interpreted as the double coset action in (10) and $\text{Fix}([U^p(m)gU^p(m)])$ as in (15).

The following definitions are motivated in two ways. On one hand, we want to allow a twist by high powers of Frobenius so that the fixed point formula is available. On the other hand, we want to separate slope components of elements in $J_b(\mathbb{Q}_p)$ in terms of $p$-adic valuation, which will play a role in harmonic analysis later.

**Definition 6.1.** An element $\delta \in J_b(\mathbb{Q}_p)$ is called acceptable if $\delta = (\delta_i)$ viewed inside $\prod_{i=1}^r \text{End}^0(\Sigma_i)\times$ verifies the following condition: if $\lambda_i > \lambda_j$ (i.e. $i < j$), any eigenvalue $e_i$ of $\delta_i$ and $e_j$ of $\delta_j$ satisfy $v_p(e_i) < v_p(e_j)$. Here $v_p : \mathbb{Q}_p^\times \to \mathbb{Q}$ is an additive $p$-adic valuation.

**Definition 6.2.** A function $\varphi \in C^\infty_c(\mathbb{A}^{\infty,p} \times J_b(\mathbb{Q}_p))$ is called acceptable if

(i) For any $(g, \delta) \in G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ in supp $\varphi$, the element $\delta$ is acceptable and belongs to $S_b$, and there exist a finite subset $I \subset G$, a sufficiently small subgroup $U^p(m)$ (in particular, having no finite torsion elements) and $(\alpha_p)_{g \in I} \in \mathbb{C}$ satisfying $\varphi = \Sigma_{g \in I} \alpha_g \text{char}_{U^p(m)gU^p(m)}$ such that

(ii) $\text{Fix}([U^p(m)gU^p(m)])$ is a finite set for every $g \in I$, and

(iii) For every $g \in I$, the formula $(16)$ holds.

We will show in Lemma 6.4 that acceptable functions are abundant enough to establish an identity of representations in $\text{Groth}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$. So it is harmless to assume that the test function $\varphi$ is acceptable when computing the trace. First we prove that any given test function becomes acceptable after enough twists.

**Lemma 6.3.** For each $\varphi \in C^\infty_c(\mathbb{A}^{\infty,p} \times J_b(\mathbb{Q}_p))$, there exists a positive integer $M$ such that whenever $N > M$, the function $\varphi^{(N)}$ defined by $\varphi^{(N)}(g) = \varphi(g \cdot (fr^s)^N)$ is acceptable.

**Proof.** The proof is easily reduced to the case $\varphi = \text{char}_{U^p(m)gU^p(m)}$ where $U^p(m)$ is small enough. From the definition of acceptable elements and the set $S_b$, there is clearly an integer $M$ such that every $\varphi^{(N)}$ for $N > M$ satisfies (i) of Definition 6.2.

The conditions (ii) and (iii) can be verified using Fujiwara’s trace formula (a.k.a. Deligne’s conjecture). For this purpose, we choose a particular model $J_{b,U^p,m}$ over some finite field $\mathbb{F}_p$, such that $J_{b,U^p,m} \times_{g^\circ} \mathbb{F}_p \simeq \mathbb{I}_{b,U^p,m}$, and $F_{ab}^s \times 1 = fr^{-s}$ under this isomorphism. Here $F_{ab}^s$ is the absolute Frobenius on $J_{b,U^p,m}$ and $fr^{-s} \in J_b(\mathbb{Q}_p)$ acts on $\mathbb{I}_{b,U^p,m}$ as described in §5. (That we can choose $J_{b,U^p,m}$ is explained in [Shi07, §2.3] in more detail. For this we assume that $\mathbb{F}_p$ contains $k(w)$ by enlarging $s$ if necessary.)

According to Fujiwara’s formula ([Fuj97, Cor 5.4.5], [Var07, Thm 2.3.2]), the following is true: there exists an integer $M' > 0$ such that whenever $N > M'$, Fix$((F_{ab}^s \times 1)^N \circ [U^p(m)gU^p(m)])$ is a finite set and the identity (16) holds with $[U^p(m)gU^p(m)]$ replaced by $(F_{ab}^s \times 1)^N \circ [U^p(m)gU^p(m)]$. The number $M'$ can be chosen independently of the coefficient sheaf.

By the identity of correspondences on $\mathbb{I}_{b,U^p,m}$

$$[U^p(m)g(fr^{-s})^N U^p(m)] = ((F_{ab}^s \times 1)^N) \circ [U^p(m)gU^p(m)],$$

the conditions (ii) and (iii) are verified by $\varphi^{(N)}$ for every $N > M'$. Finally increase $M$, if necessary, to ensure $M \geq M'$.

\[\square\]
Lemma 6.4. Suppose that $\Pi_1$ and $\Pi_2$ belong to $\text{Groth}(G(\mathbb{A}^\infty_p) \times J_b(\mathbb{Q}_p))$. If $\text{tr} \; \Pi_1(\varphi) = \text{tr} \; \Pi_2(\varphi)$ for every acceptable function $\varphi$, then $\Pi_1 \simeq \Pi_2$ in $\text{Groth}(G(\mathbb{A}^\infty_p) \times J_b(\mathbb{Q}_p))$.

Proof. For simplicity of notation, let $H := G(\mathbb{A}^\infty_p) \times J_b(\mathbb{Q}_p)$. Let us choose an arbitrary function $\varphi \in C_c^\infty(H)$ (which is not necessarily acceptable). If we show $\text{tr} \; \Pi_1(\varphi) = \text{tr} \; \Pi_2(\varphi)$, then the proof will be complete.

There are only finitely many irreducible representations $\{\pi_i\}_{i \in I}$ of $H$ contributing to $\Pi_1$ or $\Pi_2$ such that $\pi_i(\varphi)$ is nontrivial. Let $m_i$ (resp. $n_i$) be the multiplicity of $\pi_i$ in $\Pi_1$ (resp. $\Pi_2$) and set

$$\Pi'_1 := \sum_{i \in I} m_i \pi_i, \quad \Pi'_2 := \sum_{i \in I} n_i \pi_i.$$ 

Set $t := fr^s$. Note that $t$ belongs to the center of $H$. Consider the map $\theta : H \times \mathbb{Z} \to H$ given by $(h, z) \mapsto (h \cdot t^z)$. Write $\Pi'_1$ and $\Pi'_2$ for pullbacks of $\Pi'_1$ and $\Pi'_2$ along $\theta$. For each $i \in I$, the pullback of $\pi_i$ by $\theta$ has the form $\pi_i \otimes \chi_i$ for a character $\chi_i$ of $\mathbb{Z}$. Define $\varphi^{(z)} \in C_c^\infty(H)$ by $\varphi^{(z)}(h) := \varphi(ht^z)$. By Lemma 6.3, there exists a constant $C > 0$ (depending only on $\varphi$) such that $\varphi^{(z)}$ is acceptable for all $z > C$. By assumption,

$$\text{tr} \; \Pi'_1(\varphi^{(z)}) = \text{tr} \; \Pi'_2(\varphi^{(z)}), \quad \forall z > C. \tag{17}$$

We claim that for any $\psi \in C_c^\infty(\mathbb{Z}_{< -C})$

$$\text{tr} \; \Pi'_1(\varphi \times \psi) = \text{tr} \; \Pi'_2(\varphi \times \psi). \tag{18}$$

Once we prove the claim, since there are finitely many characters $\{\chi_i\}_{i \in I}$ (not necessarily distinct), it follows that (18) is true for any $\psi \in C_c^\infty(\mathbb{Z})$. In particular, we choose $\psi$ to be a function supported on $0 \in \mathbb{Z}$ to deduce that $\text{tr} \; \Pi'_1(\varphi) = \text{tr} \; \Pi'_2(\varphi)$, or $\text{tr} \; \Pi_1(\varphi) = \text{tr} \; \Pi_2(\varphi)$.

It remains to prove the above claim. It suffices to prove that (18) holds for every $\psi_y(y < -C)$ such that $\psi_y(y)$ equals 1 if $z = y$ and 0 if $z \neq y$. For any $w$ in the representation space of $\Pi'_1$, computation with respect to a Haar measure on $H$ shows

$$\Pi'_1(\varphi \times \psi_y)w = \sum_{z \in \mathbb{Z}} \int_H (\varphi(h)\psi_y(z)) \cdot \Pi'_1(h, z)w \cdot dh = \int_H \varphi(h)\Pi'_1(h^y)w \cdot dh = \int_H \varphi(ht^{-y})\Pi'_1(h)w \cdot dh = \Pi'_1(\varphi^{(-y)}) \cdot w.$$

Combining with (17), we deduce that (18) holds for $\psi = \psi_y$. This proves our claim. \hfill \square

7 $\tilde{F}_p$-points of Igusa varieties

In order to describe the set of fixed points on Igusa varieties under correspondences, we give here a moduli-theoretic description of $\tilde{F}_p$-points on Igusa varieties.

As $\tilde{Ig}_b$ has a moduli interpretation, we can describe its $\tilde{F}_p$-points in terms of abelian varieties over $\tilde{F}_p$ with additional structure. We can see from the construction of $\tilde{Ig}_b$ in §5 that $\tilde{Ig}_b(\tilde{F}_p)$ is identified with the following set $\tilde{Ig}_b^p$.

$$\tilde{Ig}_b^p = \{(A, \lambda, i, \eta^p, \{j_i\})/\sim\},$$

where
• $A$ is an abelian variety over $\mathbb{F}_p$ such that there exists an isomorphism $A[p^\infty] \cong \oplus_{i=1}^r gr^i A[p^\infty]$.

• $\lambda : A \to A^\vee$ is a prime-to-$p$ polarization.

• $i : \mathcal{O}_B \to \text{End}(A) \otimes_\mathbb{Z} \mathbb{Z}_p$ is a map of $\mathbb{Z}_p$-algebras such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$, $\forall b \in \mathcal{O}_B$.

• $\eta^p : V \otimes_\mathbb{Q} \mathbb{A}^\infty \setminus \mathbb{A}^p \cong V^p A$ is an isomorphism of $B \otimes_\mathbb{Q} \mathbb{A}^\infty \setminus \mathbb{A}^p$-modules sending $(\cdot, \cdot)$ to the $\lambda$-Weil pairing up to $(\mathbb{A}^\infty \setminus \mathbb{A}^p)^{\times}$-multiple.

• $\{\tilde{b} \}_{1 \leq i \leq r} : \Sigma_i \to gr^i A[p^\infty]$ is an isomorphism in the category $BT_{\mathbb{F}_p}^G$. (i.e. preserving $\mathcal{O}_B \otimes_\mathbb{Z} \mathbb{Z}_p$-actions and polarizations, the latter up to $\mathbb{Z}_p^{\times}$-multiple.)

• $(A, \lambda, i, \eta^p, \{j_i\})$ and $(A', \lambda', i', \eta'^p, \{j'_i\})$ are equivalent if there is a prime-to-$p$ isogeny $A \to A'$ sending $(\lambda, i, \eta^p, \{j_i\})$ to $(\gamma, i, \eta'^p, \{j'_i\})$ where $\gamma \in \mathbb{Z}_p^{\times}$.

We will see that the set $\tilde{I}_p^G$ is in natural bijection with the following set $\tilde{I}_G$ which is simpler to describe. Note that the prime-to-$p$ condition is removed below.

$$\tilde{I}_G = \{(A, \lambda, i, \eta^p, \{j_i\}) / \sim, \text{ where}$$

• $A$ is an abelian variety over $\mathbb{F}_p$.

• $\lambda : A \to A^\vee$ is a polarization.

• $i : B \to \text{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$ is a map of $\mathbb{Q}$-algebras such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$, $\forall b \in B$.

• $\eta^p : V \otimes_\mathbb{Q} \mathbb{A}^\infty \setminus \mathbb{A}^p \cong V^p A$ is an isomorphism of $B \otimes_\mathbb{Q} \mathbb{A}^\infty \setminus \mathbb{A}^p$-modules sending $(\cdot, \cdot)$ to the $\lambda$-Weil pairing up to $(\mathbb{A}^\infty \setminus \mathbb{A}^p)^{\times}$-multiple.

• $\{\tilde{b} \}_{1 \leq i \leq r} : \Sigma_i \to gr^i A[p^\infty]$ is a quasi-isogeny, which is an isomorphism in the category $BT_{\mathbb{F}_p}^G$. (i.e. preserving $B$-actions and polarizations, the latter up to $Q^{\times}_p$-multiple.)

• $(A, \lambda, i, \eta^p, \{j_i\})$ and $(A', \lambda', i', \eta'^p, \{j'_i\})$ are equivalent if there is an isogeny $A \to A'$ sending $(\lambda, i, \eta^p, \{j_i\})$ to $(\gamma\lambda, i', \eta'^p, \{j'_i\})$ where $\gamma \in Q^{\times}$.

**Lemma 7.1.** (Compare [HT01, Lem V.1.1.1])

There is a natural bijection between $\tilde{I}_G(\mathbb{F}_p)$ and $\tilde{I}_G^p$. The natural map $\tilde{I}_G^p \to \tilde{I}_G$ is also a bijection.

**Proof.** The first sentence is clear from the moduli description of the variety $I_G$. We will prove the second sentence of the lemma.

We first prove that the map $\tilde{I}_G^p \to \tilde{I}_G$ is injective. In other words, if $(A, \lambda, i, \eta^p, \{j_i\})$ and $(A', \lambda', i', \eta'^p, \{j'_i\})$ in $\tilde{I}_G^p$ become equivalent in $\tilde{I}_G$ by an isogeny $f : A \to A'$, then we need to find a prime-to-$p$ isogeny which identifies the two data in $\tilde{I}_G^p$. But $f$ itself has to be a prime-to-$p$ isogeny since $j'_i = f \circ j_i$ for every $i$ where both $j_i$ and $j'_i$ are isomorphisms. Also if $\gamma \in Q^{\times}$ is such that $\lambda$ is sent to $\gamma\lambda$, then $\gamma$ should belong to $Z_{(p)}^*$ since both $\lambda$ and $\lambda'$ are prime-to-$p$ polarizations. Therefore $f$ gives an equivalence in $\tilde{I}_G^p$.

We prove the map is surjective. We start from an element $(A, \lambda, i, \eta^p, \{j_i\})$ in $\tilde{I}_G$ and obtain an element in $\tilde{I}_G^p$ using the equivalence in $\tilde{I}_G$. By changing $A$ by an isogeny if necessary, we may assume that the condition $A[p^\infty] \cong \oplus_{i=1}^r gr^i A[p^\infty]$ is satisfied. Keeping the last condition, all $j_i$ can be arranged to be isomorphisms. We explain the last point in more detail. First we may assume that each quasi-isogeny $(j^{-1}_i)$ is an isogeny by applying $p$-power multiplication map to $A$ if necessary. Let $H_i := \ker j^{-1}_i$, $A := A/(\oplus_i H_i)$ and $f : A \to A/(\oplus_i H_i)$ be the natural quotient map. Then $f$ sends $(j_i)$ to $f \circ j_i$, but $f \circ j_i : \Sigma_i \to gr^i A[p^\infty]$ is an isomorphism for each $i$ by construction.
Next we want $\lambda$ to be prime-to-$p$. This is easy because the equivalence in $\tilde{\text{Ig}}_b$ allows multiplying a scalar in $Q^\times_p$ to $\lambda$. First observe that the following diagram commutes for some $\gamma \in Q^\times_p$ where maps are allowed to be quasi-isogenies.

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{j} & A[p^\infty] \\
\downarrow{\lambda} & & \downarrow{\gamma} \\
\Sigma' & \xrightarrow{j'} & A'[p^\infty]
\end{array}
\]

Write $\gamma = p^a u$ for $a \in \mathbb{Z}$ and $u \in Z^\times_p$. Then we simply replace $\lambda$ with $p^a \lambda$ to get a prime-to-$p$ polarization. (Recall that $\lambda_{\Sigma'}$ is already a prime-to-$p$ polarization.) At this point, it only remains to check that the image of $O_B$ under $i$ lies in $\text{End}(A) \otimes \mathbb{Z}(p)$, but this is automatic since $O_B$ is a maximal $\mathbb{Z}(p)$-order of $B$. Now our new $(A, \lambda, i, \eta, \{j_i\})$ belongs to $\tilde{\text{Ig}}_b^p$, completing the proof of surjectivity.

In [Man05, §5], it was shown that the action of $G(A^{\infty,p}) \times S_b$ on $\tilde{\text{Ig}}_b^p$ extends to an action of $G(A^{\infty,p}) \times J_b(Q_p)$. When this action is transported to $\tilde{\text{Ig}}_b$ via the bijection in Lemma 7.1, the action of each element $(\alpha, \beta) \in G(A^{\infty,p}) \times J_b(Q_p)$ on $\tilde{\text{Ig}}_b$ can be described as

\[(A, \lambda, i, \eta, \{j_i\}) \mapsto (A, \alpha \lambda, i, \eta^p \circ \alpha, \{j_i \circ \beta\}).\]

In view of Lemma 7.1, the right $G(A^{\infty,p}) \times J_b(Q_p)$-set $\tilde{\text{Ig}}_b(F_p)$ will be described in terms of $\tilde{\text{Ig}}_b$ from now on. To further analyze $\tilde{\text{Ig}}_b(F_p)$, we consider the fibration of this set over the set of the triples $(A, \lambda, i)$.

**Definition 7.2.** We define the set $\text{PIC}_b = \{(A, \lambda, i)/ \sim \}$ whose representatives are those $(A, \lambda, i)$ that appear in the description of $\tilde{\text{Ig}}_b(F_p)$ (i.e. $\exists \eta, \{j_i\}$ such that $(A, \lambda, i, \eta, \{j_i\}) \in \tilde{\text{Ig}}_b(F_p)$). We consider $(A, \lambda, i)$ and $(A', \lambda', i')$ equivalent if there is an isogeny $A \to A'$ sending $\lambda$ and $i$ to $\gamma \lambda'$ and $i'$ for some $\gamma \in Q^\times$.

By construction, we have a natural $G(A^{\infty,p}) \times J_b(Q_p)$-equivariant (with trivial action on $\text{PIC}_b$) surjection of sets

\[\pi : \tilde{\text{Ig}}_b(F_p) \to \text{PIC}_b \text{ defined by } (A, \lambda, i, \eta, \{j_i\}) \mapsto (A, \lambda, i).\]

Before we give a group theoretic expression of the fibers of $\pi$, we set up some notation. Let $z = [(A, \lambda, i)]$ be an equivalence class in $\text{PIC}_b$. We define the following.

- $C_{(A, \lambda, i)} := \text{End}_b^0(A)$,
- $M_{(A, \lambda, i)} := Z(C_{(A, \lambda, i)})$
- $\tau_{(A, \lambda, i)}$ is the Rosati involution $f \mapsto \lambda^{-1} f^\vee \lambda$ on $C_{(A, \lambda, i)}$,
- $H_{(A, \lambda, i)}$ is the $Q$-group scheme such that $H_{(A, \lambda, i)}(R) := \{g \in C_{(A, \lambda, i)} \otimes Q R| \det \tau_{(A, \lambda, i)} \in R^\times\}$

Suppose that $[(A, \lambda, i)] = [(A', \lambda', i')]$ and let $f : A \to A'$ be an isogeny providing the equivalence of triples. Then the induced identification $M_{(A, \lambda, i)} = M_{(A', \lambda', i')}$ is independent of the choice of $f$. But $f$ induces an isomorphism of the pairs $(C_{(A, \lambda, i)}, \tau_{(A, \lambda, i)})$ and $(C_{(A', \lambda', i')}, \tau_{(A', \lambda', i')})$ and an isomorphism of $Q$-groups $H_{(A, \lambda, i)} \cong H_{(A', \lambda', i')}$ which are canonical only up to $H_{(A, \lambda, i)}(Q)$-conjugacy. Keeping this in mind, we may sometimes write $C_{(A, \lambda, i)}$, $M_{(A, \lambda, i)}$, $\tau_{(A, \lambda, i)}$ and $H_{(A, \lambda, i)}$ as $C_z$, $M_z$, $\tau_z$ and $H_z$ when $H_{(A, \lambda, i)}(Q)$-conjugacy is harmless.
Let us give an embedding \( \iota_{(A,\lambda,i)} : H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p) \). For this we need to choose some \((\eta_0^i, \{j_0,i\})\) which defines a point in \( \text{Ig}_b(\mathbb{F}_p) \) together with \((A,\lambda,i)\). First consider the composite map

\[
\text{End}^0_B(A) \otimes \mathbb{A}^\infty \hookrightarrow \text{End}_B(V^pA) \cong \text{End}_B(V \otimes \mathbb{A}^\infty)
\]

where the latter isomorphism is \( g \mapsto (\eta_0^i)^{-1}g\eta_0^i \). Thereby we get an embedding of groups

\[
H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^\infty).
\]

On the other hand, we have maps

\[
\text{End}^0_B(A) \otimes \mathbb{Q}_p \hookrightarrow \text{End}^0_B(A[p^\infty]) \cong \text{End}^0_B(\Sigma)
\]

where the second map is induced from the quasi-isogeny \( \Sigma \to A[p^\infty] \) given by \( \{j_0,i\} \). By restricting to the elements preserving polarizations on both sides, obtain

\[
H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow J_b(\mathbb{Q}_p).
\]

Putting these together, we obtain an embedding \( \iota_{(A,\lambda,i)} : H_{(A,\lambda,i)}(\mathbb{A}^\infty) \hookrightarrow G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p) \), which is canonical up to \( G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p) \)-conjugacy. The next lemma, whose proof is straightforward, gives a description of the fibers of \( \pi \).

**Lemma 7.3.** See [HT01, Lem V.1.2]

Choose of a base point \( \bar{x} \) in \( \pi^{-1}[(A,\lambda,i)] \). This gives a bijection of sets with right \( G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p) \)-action

\[
\pi^{-1}[(A,\lambda,i)] \cong \iota_{(A,\lambda,i)}(H_{(A,\lambda,i)}(\mathbb{Q}_p)) \setminus (G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p))
\]

If we choose \( \bar{x}' = \bar{x}h \) \((h \in G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p)) \) as a base point, the above isomorphism changes by multiplication by \( h \) while \( \iota_{(A,\lambda,i)} \) changes by conjugation by \( h \).

Suppose that \( g \) is an element of \( G(\mathbb{A}^\infty) \times S_b \) and that \([U^p(m)gU^p(m)]\) satisfies the conditions (ii) and (iii) of Definition 6.2. For simplicity let us write \( U \) for \( U^p(m) \) in this section. We deduce from (11) and (16) that

\[
\text{tr}(\text{char}_{UgU}|_{H_c(\text{Ig}_b, \mathbb{L}_\xi)}) = \text{vol}(U) \sum_{x \in \text{Fix}([UgU])} \text{tr}((UgU)|(\mathbb{L}_\xi)_{\text{tr}})
\]

The sum has finitely many nonzero terms and is finite. Our next task is to analyze the set \( \text{Fix}([UgU]) \), which is given by (15). Let us write \( G_b \) for \( G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p) \). Arguing as in [HT01, p.153-155], the expression in (15) can be rewritten as

\[
\text{Fix}([UgU]) = \coprod_{z \in \text{PIC}_a} \coprod_{a \in H_4(\mathbb{Q})/\sim} \iota_{z}(H_{z}(\mathbb{Q}))\backslash\{y \in G_b|y^{-1}\iota_z(a)y \in gU\}/U \cap gUg^{-1}
\]

where the equivalence relation in \( H_z(\mathbb{Q}) \) is given by \( H_z(\mathbb{Q}) \)-conjugacy action. Again proceeding as in [HT01, Lem V.1.4], we obtain the first form of the counting point formula.

**Lemma 7.4.** Suppose that \( \varphi \in C_0^\infty(G(\mathbb{A}^\infty) \times J_b(\mathbb{Q}_p)) \) is an acceptable function. Then

\[
\text{tr}(\varphi|_{H_c(\text{Ig}_b, \mathbb{L}_\xi)}) = \sum_{z \in \text{PIC}_a} \sum_{a \in H_4(\mathbb{Q})/\sim} \text{vol}(\iota_{z}(Z_{H_z}(a)(\mathbb{Q}))\backslash Z_{G_{b}}(\iota_{z}(a))) \cdot O_{z(a)}^{G_{b}}(\varphi) \cdot \text{tr}(\xi_{z}(a))
\]

The sum has finitely many nonzero terms and is finite. The measure on \( \iota_{z}(Z_{H_z}(a)(\mathbb{Q})) \) is chosen such that every point has measure 1. Haar measures on other groups are chosen to be compatible with each other.
Remark 7.5. Since the group $H_2(\mathbb{R})$ is compact modulo center, any element $a \in H_2(\mathbb{Q})$ is semisimple and elliptic in $H_2(\mathbb{R})$.

8 Honda-Tate theory

In this section we use a version of Honda-Tate theory to parametrize the pairs $(A, i)$ by $p$-adic types over $F$. We also give a necessary condition for $(A, i)$ to appear in the set $\text{PIC}_b$. We generalize the notion of $p$-adic types in [HT01, V.2] in order to classify isogeny classes of abelian varieties over $\mathbb{F}_p$ which are not necessarily simple. Before defining $p$-adic types, we will set up some notation.

- Let $I$ be a finite index set.
- Let $M_t$ be a CM field or a totally real field for each $t \in I$. Note that $M_t$ has a well-defined complex conjugation $c$, an automorphism of order 2 or 1, respectively.
- Let $\mathfrak{P}_M$ be the set of places of $M$ over $p$, also written as $\mathfrak{P}_t$ for simplicity.
- $\mathbb{Q}[\mathfrak{P}_t] := \oplus_{x \in \mathfrak{P}_t} \mathbb{Q} \cdot x$ is the $\mathbb{Q}$-vector space with basis $\mathfrak{P}_t$.
- If $\mathfrak{a}$ is a fractional ideal of $M_t$, we define $[\mathfrak{a}] := \sum_{x \in \mathfrak{P}_t} x(\mathfrak{a}) \cdot x \in \mathbb{Q}[\mathfrak{P}_t]$.
- If $i : M \hookrightarrow N$ is a finite extension of fields where $M, N$ are totally real or CM, we define a $\mathbb{Q}$-linear map $i_* : \mathbb{Q}[\mathfrak{P}_M] \to \mathbb{Q}[\mathfrak{P}_N]$ by $x \mapsto \sum_{y|x} c_y/y$.

Definition 8.1. Let $F_0$ be a number field. A $p$-adic type over $F_0$ is a quadruple $(M, \vec{\eta}, \vec{n}, \kappa)$ where

- $M = \prod_{t \in I} M_t$ is a product of totally real or CM fields for a nonempty index set $I$,
- $\vec{\eta} = (\eta_t)_{t \in I}$ where $\eta_t = \sum_{x \in \mathfrak{P}_t} \eta_{t,x} x \in \mathbb{Q}[\mathfrak{P}_t]$,
- $\vec{n} = (n_t)_{t \in I}$ is a collection of positive integers and
- $\kappa : F_0 \rightarrow M$ is a $\mathbb{Q}$-algebra homomorphism.

such that for all $t \in I$, $\eta_t + c_t \eta_t = [p]$ and $\forall x \in \mathfrak{P}_t$, $\eta_{t,x} \geq 0$. We will often drop $\kappa$ from the data when $\kappa$ is well understood as the $F_0$-algebra structure map of $M$.

Definition 8.2. A $p$-adic type $(M, \vec{\eta}, \vec{n}, \kappa)$ is called simple if $M$ is a field and $\vec{n} = (1)$. Such a $p$-adic type will often be written as $(M, \eta)$ when $\kappa$ is understood.

Remark 8.3. The $p$-adic types defined in [HT01, V.2] correspond to our simple $p$-adic types.

We say $(M', \vec{\eta}', \vec{n}', \kappa')$ and $(M'', \vec{\eta}'', \vec{n}'', \kappa'')$ are equivalent over $F_0$ if there exist a $p$-adic type $(M, \vec{\eta}, \vec{n}, \kappa)$ and $F_0$-algebra embeddings $i' : M' \hookrightarrow M$, $i'' : M'' \hookrightarrow M$ ($F_0$-structure given by $\kappa, \kappa', \kappa''$) such that

(i) Whenever $t'_1 \neq t'_2$ and $t''_1 \neq t''_2$, we have $i'(M'_{t'_1})i'(M'_{t'_2}) = 0$ and $i''(M''_{t''_1})i''(M''_{t''_2}) = 0$.

(ii) There is a partition of the index set $I' = \bigsqcup_{t \in I} I'_t$ for $(M', \vec{\eta}', \vec{n}')$ satisfying the following: for all $t' \in I'_t$, $i'$ induces $M'_{t'} \hookrightarrow M_t$, $i'_* \eta_{t'} = \eta_t$, and $\sum_{t' \in I'_t} n_{t'} = n_t$. There is a partition $I'' = \bigsqcup_{t \in I} I''_t$ such that an exactly analogous condition holds for $(M'', \vec{\eta}'', \vec{n}'')$.

Two $p$-adic types $(M', \vec{\eta}', \vec{n}', \kappa')$ and $(M'', \vec{\eta}'', \vec{n}'', \kappa'')$ over $F_0$ are said to be isomorphic if there exists an $F_0$-algebra isomorphism $M' \xrightarrow{\sim} M''$ sending $\vec{\eta}', \vec{n}'$ to $\vec{\eta}'', \vec{n}''$.

We can define an $F_0$-minimal representative of any equivalence class of $p$-adic types over $F_0$. We begin with simple $p$-adic types first. A simple $p$-adic type $(M, \eta)$ over $F_0$ is minimal if for any
other simple $p$-adic type $(M', \eta')$ equivalent to $(M, \eta)$, there exists an $F_0$-algebra homomorphism $i' : M \to M'$ such that $\eta' = i'_* \eta$. It can be easily seen that any equivalence class of simple $p$-adic types over $F_0$ has a minimal representative: we can pushforward a given $p$-adic type into any big CM field $M$ which is Galois over $F_0$ to get $(M, \tilde{\eta})$ and take the fixed field $M$ of $M$ under Galois automorphisms preserving the $\eta$. Then the descended $p$-adic type $(M, \eta)$ is $F_0$-minimal.

Now we consider $p$-adic types that are not necessarily simple. We say that a $p$-adic type $(M, \eta, \tilde{\eta})$ is \textit{minimal} over $F_0$ if every constituent $(M_i, \eta_i)$ is minimal as a simple $p$-adic type over $F_0$ and no two constituents $(M_{i_1}, \eta_{i_1})$ and $(M_{i_2}, \eta_{i_2})$ are equivalent over $F_0$ for $i_1 \neq i_2$. It is easy to check that the new definition of minimality coincides with the one for simple $p$-adic types and that there exists a unique minimal representative over $F_0$ up to isomorphism in any equivalence class of $p$-adic types.

We are about to explain a version of Honda-Tate theory. One may see [HT01, p.158-159] for more detail and other references in the case of simple $p$-adic types over $M$ and $Q$ is equipped with $I$-equivalence classes of simple $AV$ whose invariants at places $p$ is Galois over $M$ and $Q$ splits by assumption.

We consider the category $AV_B^0$ whose objects are the pairs $(A, i)$ where $A$ is an abelian variety over $\overline{F}_p$ and $i : B \to \text{End}_B^0(A)$ is a $\overline{F}_p$-algebra homomorphism. The morphisms from $(A, i)$ to $(A', i')$ are elements $f \in \text{Hom}(A, A') \otimes \mathbb{Z}$ such that $f \circ i(b) = i'(b) \circ f$ for all $b \in B$. We denote by $\text{End}_B^0(A, i)$ or $\text{End}_B^0(A)$ an endomorphism algebra in $AV_B^0$. By an easy extension of the Poincaré reducibility theorem, $AV_B^0$ is an $F$-linear semisimple category. We classify simple objects of $AV_B^0$ using [Kot92, §2]. If $(A, i)$ is a simple object, then $A$ is isogenous to $A_0^n$ for a simple abelian variety $A_0$ over $\overline{F}_p$ and $n \in \mathbb{Z}_{>0}$. Moreover, the centralizer of $B$ in $M_m(\text{End}^0(A_0))$ is a division algebra, whose center is denoted $M$. Let $(M_0, \eta_0)$ be the minimal simple $p$-adic type (over $\overline{Q}$) associated to $A_0$. The field $M$ is equipped with $\overline{Q}$-algebra maps $j : M_0 \to M$ and $\kappa : F \to M$ coming from $i$. Then

$$\begin{align*}
(A, i) & \mapsto (M, j_* \eta_0, (1), \kappa)
\end{align*}$$

is how we associate a simple $p$-adic type over $F$ to each simple object $(A, i)$ of $AV_B^0$. This induces a well-defined bijection between the set of isomorphism classes of simple objects of $AV_B^0$ and the set of equivalence classes of simple $p$-adic types over $F$.

As $AV_B^0$ is a semisimple category, any object $(A, i)$ is isomorphic to $\oplus_{t \in I} (A_t, i_t)^{n_t}$ for a finite set $I$ and simple objects $(A_t, i_t)$ such that there is no nontrivial morphism between $(A_t, i_t)$ and $(A_{t'}, i_{t'})$ for $t \neq t'$. If $(A_t, i_t)$ corresponds to $(M_t, \eta_t, (1), \kappa_t)$ for each $t \in I$, we define the following association:

$$\begin{align*}
(A, i) & \mapsto \left( \prod_{t \in I} M_t, (\eta_t)_{t \in I}, (n_t)_{t \in I}, (\kappa_t)_{t \in I} \right).
\end{align*}$$

We construct from a simple minimal $p$-adic type $(M_t, \eta_t, \kappa_t)$ a division algebra $C_t$ with center $M_t$ whose invariants at places $x$ of $M_t$ are given by the following. (Recall that $B \otimes \overline{Q} \mathbb{Q}_p$ splits by assumption.)
\[
\text{inv}_x(C_t) = \begin{cases} 
1/2 - \text{inv}_x(B \otimes F M_{t,x}), & x \text{ real} \\
\eta_x f_{x/p}, & x\mid p, x \nmid \infty \\
-\text{inv}_x(B \otimes F M_{t,x}), & x \mid p, x \nmid \infty 
\end{cases}
\] (23)

**Proposition 8.4.** The map (22) gives a natural bijection between the following two sets.

(i) The set of isomorphism classes in \(AV_B^0\).

(ii) The set of equivalence classes of \(p\)-adic types over \(F\).

We can find a minimal representative \((\prod_{t \in I} M_t, (\eta_t)_{t \in I}, (\kappa_t)_{t \in I}, (\kappa_t)_{t \in I})\) corresponding to \(\oplus_{t \in I}(A_t, i_t)^n_t\), where \((A_t, i_t)\) are simple objects in distinct isomorphism classes of \(AV_B^0\), such that the following are true for each \(t \in I\).

- \(M_t = Z(\text{End}^0(A_t, i_t))\) and \(C_t \simeq \text{End}^0(A_t, i_t)\).
- \(A_t[x^\infty]\) has pure slope \(\eta_t/e_{x/p}\) for each place \(x\) of \(M_t\) over \(p\), and height \([M_{t,x} : \mathbb{Q}_p][B : F]^{1/2}[C_t : M_t]^{1/2}\).

**Proof.** The bijection between (i) and (ii) is straightforward given the discussion preceding this proposition. The assertions about \(M_t, C_t\) and the slope follow from the case of simple \(p\)-adic types over \(\mathbb{Q}\) using general facts in [Kot92, §3].

Let \(v\) be a place of \(F\) over \(p\) and \((M_t, \eta_t)\) a simple \(p\)-adic type over \(F\). For \(\lambda \in \mathbb{Q}\), we define \(S_{\lambda,v}(M_t)\) to be the places \(x\) of \(M_t\) over \(v\) such that \(\lambda = \eta_t/e_{x/p}\). Recall that the BT-group \(\Sigma = \bigoplus_{i \in I} \Sigma^i\) has an action by \(\mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_p \simeq \prod_{v \mid p} \mathcal{O}_{F,v}\). Correspondingly we have a decomposition \(\Sigma^i = \bigoplus_{v \mid p} \Sigma^i[v^\infty]\) for each \(i\).

**Corollary 8.5.** The bijection in Proposition 8.4 restricts to the bijection of the following two sets.

(i) The set of isomorphism classes of \((A, i)\) in \(AV_B^0\) for which there exists a quasi-isogeny \(j : \Sigma \to A[p^\infty]\) compatible with the action of \(B_{Q_p}\).

(ii) The set of equivalence classes of \(p\)-adic types \((\prod_{t \in I} M_t, (\eta_t)_{t \in I}, (\kappa_t)_{t \in I})\) over \(F\) such that

- For each \(\lambda \in \mathbb{Q}\), there exist \(t \in I\) and a place \(v \mid p\) of \(F\) such that \(S_{\lambda,v}(M_t) \neq \emptyset\) if and only if there exists \(i(1 \leq i \leq r)\) such that \(\lambda = \lambda_i\) (i.e., \(\lambda\) is among the slopes of \(\Sigma\)).
- For each \(1 \leq i \leq r\), \(\lambda = \lambda_i\) and each place \(v \mid p\) of \(F\),

\[\sum_{t \in I} \sum_{x \in S_{\lambda,v}(M_t)} n_t[M_{t,x} : \mathbb{Q}_p][B : F]^{1/2}[C_t : M_t]^{1/2} = \text{height}(\Sigma^i[v^\infty]).\]

**Proof.** Given Proposition 8.4, we only need to check that the additional conditions in (i) and (ii) match. Since \(B \otimes_F F_v\) is a matrix algebra over \(F_v\), the problem boils down to comparing the slope decompositions of \(\Sigma[v^\infty]\) and \(A[v^\infty]\). The first condition in (ii) means that \(\Sigma\) and \(A[p^\infty]\) have the same set of slopes. The second condition in (ii) implies that the heights of each slope component are the same for \(\Sigma[v^\infty]\) and \(A[v^\infty]\).

The following lemma, generalizing [HT01, Lem V.2.2], is indispensable in later argument.
Lemma 8.6. Let $z = \{ (A, \lambda, i) \}$ be an equivalence class in $\text{PIC}_k$. Let $\iota_z(a) = (a^p, a_p)$ be the image of $a \in H_2(\mathbb{Q})$ in $G(\mathbb{A}^{\infty-p}) \times J_0(\mathbb{Q}_p)$. Suppose that $a_p$ is acceptable. Then $M_z \subset F(a)$ (as $F$-subalgebras in $\text{End}_F^0(A)$) and
\[
Z_{G(\mathbb{A}^{\infty-p}) \times J_0(\mathbb{Q}_p)}(\iota_z(a)) = \iota_z((ZH_1(a)(\mathbb{A}^{\infty-p})))
\]
(24)

Proof. We explain how the first assertion implies the second assertion. Clearly the inclusion $\supset$ always holds in (24). In case $M_z \subset F(a)$, we have the other inclusion. Indeed, if $g \in G(\mathbb{A}^{\infty-p}) \times J_0(\mathbb{Q}_p)$ centralizes $\iota_z(a)$ then $g$ centralizes $F(a)$ so it also centralizes $M_z$ (viewed inside $\text{End}_F(V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty-p}) \times \text{End}_F^0(\Sigma))$, therefore $g$ belongs to $H_2(\mathbb{A}^{\infty-p})$.

We will prove the first assertion in several steps. Let $(A, i) = (\otimes_{i=1}^n (A_i, i_n^a_i))$ where $(A_i, i_n)$ are simple objects and no two of them are isomorphic in $AV_B^0$. The pair $(A, i)$ corresponds to a minimal $p$-adic type $(\prod_{i=1}^n M_i, (\tilde{\eta}_i)_{i \in I}, (\tilde{\kappa}_i)_{i \in I})$ over $F$ so that no two $(M_i, \tilde{\kappa}_i)$ are isomorphic over $F$. Note that $M_z = \prod_{i \in I} M_i$. Write $a = (a_i) \in \prod_{i \in I} M_{\tilde{\kappa}_i}(\text{End}^1(A_i, i_i))$. The proof of $M_z \subset F(a)$ reduces to showing $M_t \subset F(a_t)$ for each $t \in I$.

We decompose $F(a_t)$ into a product of fields $F(a_t) = \prod_t F_{i_t}$. We have $F_{i, t} = F_{i, t} \otimes_F M_i = \prod_{i, j} M_{i, j}$ where $M_{i, j}$ are fields. To prove $M_t \subset F(a_t)$, it suffices to prove that $M_t \subset F_{i, t}$ in $M_{i, j}$ for all $i, j$. In the following, we work with fixed $i, j$ and write $M, \tilde{M}$ for $M_t, \tilde{M}$ for $M_t, \tilde{M}_{i, j}$.

We introduce some notation. Let $S := \{ \lambda_1, \ldots, \lambda_r \}$ be the set of all slopes of the BT-group $\Sigma$ (or equivalently, $A[p^\infty]$). Let $N$ be the Galois closure of $\tilde{M} \cap M$ (the intersection taken in $A$). Let $u$ denote a place of $\tilde{M} \cap M$ over $p$. Define $\Psi_u(K)$ to be the set of places of $K$ over $u$ where $K$ is either $M, \tilde{M}$, $F$, or $N$. We have obvious maps $\Psi_u(N) \to \Psi_u(M) \to \Psi_u(M)$ by restriction of places.

Write $a_p \in J_0(\mathbb{Q}_p)$ as $a_p = (a_i)_{i=1}^r = \prod_{i=1}^r \text{End}^1(\Sigma)$ via $J_0(\mathbb{Q}_p) \subset \text{End}^1(\Sigma)$. Since $a_p$ is acceptable, we may find positive real numbers $\varepsilon_0, \ldots, \varepsilon_r$ such that for any $i \ (1 \leq i \leq r)$ and any eigenvalue $e_i$ of $a_i$, the inequality $e_i - 1 < |e_i|_p < e_i$ holds. Then we define a map $s_{\tilde{M}, u} : \Psi_u(\tilde{M}) \to S$ by $v \mapsto \lambda_i$ if $i$ is such that $e_{i-1} < |a_i|_p^{1/|F_{i, p}|} < e_i$. We also define a map $s_{M, u} : \Psi_u(M) \to S$ by $w \mapsto \eta_u/\varepsilon_w \eta_p$. This means that $A_{[w^{\infty}]}$ has pure slope $s_{M, u}(w)$ by Honda-Tate theory. We induce maps $s_{\tilde{M}, u} : \Psi_u(\tilde{M}) \to S$ and $s_{M, u} : \Psi_u(N) \to S$ from $s_{M, u}$. On the other hand, we induce a map $s_{\tilde{M}, u}' : \Psi_u(\tilde{M}) \to S$ from $s_{\tilde{M}, u}$.

Our first claim is that $s_{\tilde{M}, u}' = s_{M, u}'. To prove this, let $x$ be a place in $\Psi_u(\tilde{M})$ and put $w := x|_{\tilde{M}}$. By definition, $s_{\tilde{M}, u}'(x) = \lambda_i$ means that $e_{i-1} < |a_i|_p^{1/|M_{i, p}|} < e_i$. This is equivalent to the fact that $a$ acts on $A[w^{\infty}]$ by an eigenvalue whose $p$-adic absolute value is between $e_{i-1}$ and $e_i$, which means that $A[w^{\infty}]$ has slope $\lambda_i$. Thus the first claim follows.

We prove our second claim that $s_{\tilde{M}, u}$ is a constant function. Observe that $\sigma \in \text{Gal}(N/\tilde{M} \cap M)$ acts on $s_{N, u}$ by $f \mapsto f \circ \sigma$. We assert that $\text{Gal}(N/\tilde{M})$ fixes $s_{N, u}$. Indeed, if $f$ is a place in $\Psi_u(N)$ such that $s_{N, u}(f) \neq s_{N, u}(\sigma f)$, then $s_{\tilde{M}, u}(y|_{\tilde{M}}) \neq s_{\tilde{M}, u}(\sigma y|_{\tilde{M}})$, but this contradicts $y|_{\tilde{M}} = \sigma y|_{\tilde{M}}$. It is obvious that $\text{Gal}(N/M)$ also fixes $s_{N, u}$. Since $\text{Gal}(N/\tilde{M} \cap M)$ is generated by $\text{Gal}(N/\tilde{M})$ and $\text{Gal}(N/M)$, we conclude that $s_{N, u}$ is fixed under $\text{Gal}(N/M, N)$, the latter group is transitive on $\Psi_u(N)$, implying that $s_{N, u}$ is a constant function. So $s_{\tilde{M}, u}$ is also a constant function.

Now we construct a simple $p$-adic type $(\tilde{M} \cap M, \xi, \kappa)$ over $F$. The map $\kappa$ is induced from $\kappa_c : F \to M$. Let $\lambda_u$ be the common image of $s_{\tilde{M}, u}$. We define $\kappa_u := \varepsilon_u/\varepsilon_p \lambda_u$ for each place $u$. Then we readily check that $(\tilde{M} \cap M, \xi, \kappa)$ is equivalent to $(M_t, \tilde{\eta}_i, \kappa_i)$ over $F$. Recall that we agreed to write $M$ for $M_t$. By the minimality of the $p$-adic type that was originally chosen, we conclude that $M \subset \tilde{M} \cap M$, namely $M \subset \tilde{F}$.

\[\square\]
9 Polarizations

This section serves as a preparatory step for parametrizing polarizations using Galois cohomology. In this section we do not assume that \((A, \lambda, i)\) represents an element of \(\text{PIC}_k\) until Definition 9.7.

We start with a general discussion of \(C\)-polarizations. Consider \((C, \ast)\) where \(C\) is a finite dimensional semisimple \(\mathbb{Q}\)-algebra whose center \(Z(C)\) is a product of CM or totally real fields and \(\ast\) is an involution on \(C\) which acts on \(Z(C)\) as complex conjugation. Let \(A\) be an abelian variety over \(\mathbb{F}_p\) and \(i : C \to \text{End}^0(A)\) be a \(\mathbb{Q}\)-algebra homomorphism. When \(\lambda : A \to A^{\vee}\) is a polarization, denote by \(\iota_{\lambda}\) the \(\lambda\)-Rosati involution on \(\text{End}^0(A)\).

**Definition 9.1.** We call a polarization \(\lambda : A \to A^{\vee}\) a \((C, \ast)\)-polarization for \((A, i)\) as above if \(\iota_{\lambda}(i(c)) = i(c^\ast)\) for all \(c \in C\). Two \((C, \ast)\)-polarizations \(\lambda_1\) and \(\lambda_2\) are equivalent if there exists \(f \in \text{End}^0_C(A)^{\times}\) such that \(\lambda_2 = \gamma f^\ast \lambda_1 f\) for some \(\gamma \in \mathbb{Q}^{\times}\). When there is no ambiguity, a \((C, \ast)\)-polarization is simply called a \(C\)-polarization.

**Lemma 9.2.** Given any \((A, i)\) as above, there exists a \(C\)-polarization \(\lambda_0\) for \((A, i)\). If the \(\mathbb{Q}\)-group \(H_{\lambda_0}\) is defined so that for any \(\mathbb{Q}\)-algebra \(R\),

\[
H_{\lambda_0}(R) = \{g \in \text{End}^0_C(A) \otimes_\mathbb{Q} R : gg^{\iota_{\lambda_0}} R^{\times}\},
\]

then the set of equivalence classes of \(C\)-polarizations for \((A, i)\) is in natural bijection with \(\ker(H^1(\mathbb{Q}, H_{\lambda_0}) \to H^1(\mathbb{R}, H_{\lambda_0}))\).

**Proof.** [Kot92, Lem 9.2] for the existence of a \(C\)-polarization. For the bijection, see [HT01, Lem V.3.1].

We go back to the notation of the previous section. From now on, whenever we consider triples \((A, \lambda, i)\) we will assume that \(\lambda\) is a \(B\)-polarization for \((A, i)\). Suppose we have two such triples \((A, \lambda, i)\) and \((A', \lambda', i')\). We say that they are \(\mathbb{Q}\)-isogenous if there is an element \(f \in (\text{Hom}(A, A') \otimes_\mathbb{Z} \mathbb{Q})^{\times}\) such that \(f \circ i'(b) = i(b) \circ f\) for all \(b \in B\) and \(f^\ast \lambda' f = \gamma \lambda\) for some \(\gamma \in \mathbb{Q}^{\times}\). A usual isogeny (or \(\mathbb{Q}\)-isogeny) of two triples is defined analogously with \(f \in (\text{Hom}(A, A') \otimes_\mathbb{Z} \mathbb{Q})^{\times}\) and \(\gamma \in \mathbb{Q}^{\times}\). Note that this notion of isogeny is the same as the equivalence relation in Definition 7.2.

For a triple \((A, \lambda, i)\), define a \(\mathbb{Q}\)-group \(H_{\lambda}\) as in (25), using \(\lambda\) and \(B\) instead of \(\lambda_0\) and \(C\). Whenever a triple \((A', \lambda', i')\) is \(\mathbb{Q}\)-isogenous to \((A, \lambda, i)\), it defines a cocycle \(\tau \mapsto f^{-1} \circ f^\ast\) in \(H^1(\mathbb{Q}, H_{\lambda})\). This association defines the bijection in the following lemma.

**Lemma 9.3.** Let \((A, i)\) be an object of \(A^B\). Choose \(\lambda_0\) as in Lemma 9.2. Then the set of isogeny classes of the triples \((A', \lambda', i')\) which are \(\mathbb{Q}\)-isogenous to \((A, \lambda_0, i)\) is in natural bijection with the set \(\ker(H^1(\mathbb{Q}, H_{\lambda_0})\to H^1(\mathbb{R}, H_{\lambda_0}))\).

**Proof.** [Kot92, Lem 17.1].

**Definition 9.4.** We say that \((A, \lambda, i)\) and \((A', \lambda', i')\) are nearly equivalent if there exists an isogeny \(f : A \to A'\) satisfying \(f \circ i(b) = i'(b) \circ f\) for all \(b \in B\) such that the map \(f\) induces

(i) an equivalence of \(V^p A\) and \(V^p A'\) as \((\ast \otimes c)\)-Hermitian \(B \otimes_F M \otimes_\mathbb{Q} A^{\infty-p}\) modules with Weil pairings given by \(\lambda, \lambda'\), respectively. Here \(M\) denotes \(M_{(\ast, \Lambda(\lambda, i))} \simeq M_{(\ast, \Lambda(\lambda', i'))}\) (identified via \(f\)).

(ii) an isogeny \([p^\infty]\) \(A'[p^\infty]\) over \(\mathbb{F}_p\) compatible with \(B \otimes_F M \otimes_\mathbb{Q} \mathbb{Q}_p\)-actions and polarizations, the latter up to \(\mathbb{Q}_p^{\ast}\)-multiple.

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We say that equivalence classes \([[(A, \lambda, i)] \text{ and } [(A', \lambda', i')]] \) are nearly equivalent if their representatives are nearly equivalent.

If \((A, \lambda, i) \) and \((A', \lambda', i') \) are nearly equivalent, then they are \(\overline{\mathbb{Q}}\)-isogenous. (One can use the same argument as in the first paragraph of page 436 of [Kot92].) So \((A', \lambda', i') \) gives an element of \(\ker(H^1(\mathbb{Q}, H_\lambda) \rightarrow H^1(\mathbb{R}, H_\lambda)) \) via Lemma 9.3.

**Lemma 9.5.** Given \((A, \lambda, i) \) as before, the set of isogeneity classes \([(A', \lambda', i')] \) that are nearly equivalent to \([(A, \lambda, i)] \) is in natural bijection with \(\ker(Q) \).

**Proof.** In the definition of near equivalence, the condition (i) means that the Galois cocycle for \([(A', \lambda', i')] \) belongs to \(\ker(H^1(\mathbb{Q}, H_\lambda) \rightarrow H^1(\mathbb{Q}, H_\lambda(\overline{\mathbb{A}}_{\infty, j}))) \) and (ii) means that the same cocycle belongs to \(\ker(H^1(\mathbb{Q}, H_\lambda) \rightarrow H^1(\mathbb{Q}_p, H_\lambda)) \). With this observation, the current lemma follows from Lemma 9.3.

\(\square\)

**Lemma 9.6.** When \([(A, \lambda, i)] \) and \([(A', \lambda', i')] \) are nearly equivalent, there is an isomorphism

\[
H_{(A,\lambda, i)} \simeq H_{(A',\lambda', i')},
\]

canonical up to conjugation by an element of \(H_{(A,\lambda, i)}(\mathbb{Q}) \).

**Proof.** The triple \((A', \lambda', i') \) defines a cocycle \(c : \tau \mapsto f^{-1} \circ f^\tau \) in \(\ker(\mathbb{Q} \to H_\lambda) \). Since \(\ker(\mathbb{Q} \to H_\lambda) \simeq \ker(\mathbb{Q}, Z(H_\lambda)) \) by [Kot92, 57], modifying \((A', \lambda', i') \) by an isogeny if necessary, we may assume that \(f^{-1} \circ f^\tau \in Z(H_\lambda \otimes Q \overline{\mathbb{Q}}) \) for all \(\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Composing \(f \) with an isogeny, we may also assume that \(f \) belongs to \((M_{(A,\lambda, i)} \otimes Q \overline{\mathbb{Q}})^\times \) since the cocycle \(c \) is trivialized under the map \(\ker(\mathbb{Q}, Z(H_\lambda)) \rightarrow H^1(\mathbb{Q}, M_{(A,\lambda, i)})) \), the latter being a trivial group by Hilbert 90.

It suffices to prove that \(\dot{\tau}_\lambda = \dot{\tau}_{\lambda'} \). This follows from the basic fact that \(f \in Z(\text{End}^0(A) \otimes Q \overline{\mathbb{Q}})^\times \) and thus \(f^{\gamma} \in Z(\text{End}^0(A^{\gamma}) \otimes Q \overline{\mathbb{Q}})^\times \). Indeed, \(\lambda = \gamma^{-1} f^{\gamma} \lambda' f \) for some \(\gamma \in Q^\times \) by near equivalence and

\[
\dot{\tau}_\lambda(g) = \lambda^{-1} g^\gamma \lambda = (f^{\gamma} \lambda' f)^{-1} g^\gamma (f^{\gamma} \lambda' f) = \lambda'^{-1} g^\gamma \lambda' = \dot{\tau}_{\lambda'}(g)
\]

\(\square\)

**Definition 9.7.** The set \(FP^AV_b \) is defined to be the set of pairs \((z, [a]) \) where

(i) \(z = [(A, \lambda, i)] \) is a near equivalence class in \(\text{PIC}_b(\mathbb{Q}) \)

(ii) \([a] \) is the \(H(z)\)-conjugacy class of \(a \in H_2(\mathbb{Q}) \), where \(a \) is an acceptable element in \(H_2(\mathbb{Q}_p) \). (We consider \(a \) acceptable if its image under the embedding \(\iota_2 : H_2(\mathbb{Q}_p) \rightarrow J_0(\mathbb{Q}_p) \) is acceptable.

This property is unchanged if \(\iota_2 \) is replaced with a \(J_0(\mathbb{Q}_p)\)-conjugate.)

**10 Kottwitz triples and Kottwitz invariants**

In this section, we define Kottwitz triples which will be used to parametrize the set \(FP^AV_b \). The Kottwitz invariant \(\alpha(\gamma_0, \gamma, \delta) \) is associated to each Kottwitz triple \((\gamma_0, \gamma, \delta) \) and tells us exactly when a Kottwitz triple arises from the moduli data of Igusa varieties.

**Definition 10.1.** By a Kottwitz triple \((\gamma_0, \gamma, \delta) \) of type \(b \), we mean a triple \((\gamma_0, \gamma, \delta) \) where

- \(\gamma_0 \in G(\mathbb{Q}) \) is semisimple, and elliptic in \(G(\mathbb{R}) \)
Lemma 10.4. The following are true.

- $\gamma \in G(\mathbb{A}^{\infty,p})$ and $\gamma_0 \sim_{\mathbb{K}^{\infty,p}} \gamma$.
- $\delta \in J_b(\mathbb{Q}_p)$ is acceptable (see Definition 6.1) and $\gamma_0 \sim_{\mathbb{K}_p} \delta$ in $G(\mathbb{Q}_p)$ via any embedding in the canonical $G(\mathbb{Q}_p)$-conjugacy class of embeddings $J_b(\mathbb{Q}_p) \hookrightarrow G(\mathbb{Q}_p)$ (as in the paragraph below Lemma 4.2).

Two Kottwitz triples $(\gamma_0; \gamma, \delta) \sim (\gamma'_0; \gamma', \delta')$ are said to be equivalent if $\gamma_0 \sim_{st} \gamma'_0$, $\gamma \sim_{\mathbb{K}^{\infty,p}} \gamma'$, and $\delta \sim \delta'$.

Remark 10.2. The notion of Kottwitz triples clearly depends on $b \in B(G, -\mu)$, but not on the extra choice of \( \tilde{b} \) in the following sense: the equivalence classes of Kottwitz triples for any two decent representatives $b$ and $b'$ of $b$ are in canonical bijection with each other via $J_b \simeq J_{b'}$.

Definition 10.3. For each $b \in B(G, -\mu)$, we define $KT_b$ to be the set of equivalence classes of all Kottwitz triples of type $b$.

We explain how to attach a $p$-adic type over $F$ to a Kottwitz triple. Denote by $F(\gamma_0)$ the $F$-algebra generated by $\gamma_0$ in $\text{End}_B(V)$. It admits a product decomposition $F(\gamma_0) \simeq \prod_{t \in I} F_t$ into fields. Let $I_t$ be the set of places of $F_t$ over $p$ so that $F_t \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{\gamma \in I_t} F_{t,y}$ is a product of fields. Since $\gamma_0 \sim_{\mathbb{K}_p} \delta$, we are able to choose an isomorphism $F_{\mathbb{Q}_p}(\gamma_0) \simeq F_{\mathbb{Q}_p}(\delta)$ such that $\gamma_0 \mapsto \delta$. Under this isomorphism, each $F_{t,y}$ is mapped nontrivially into $\text{End}_B(\Sigma^{k(t,y)})$ for only one $k(t,y) \in \mathbb{Z}$ ($1 \leq k(t,y) \leq r$). Since $\delta$ is acceptable, $k(t,y)$ is independent of the choice of the isomorphism. For each $t \in I$, we define a simple $p$-adic type $(F_t, \eta_t, \tilde{\kappa}_t)$ over $F$ by $\tilde{\eta}_{t,y} = c_{y/p}^{-1} \lambda^{-1}(t,y)$. The map $\tilde{\kappa}_t : F \to F_t$ is the $F$-algebra structure map of $F_t$. Finally we find a minimal representative $(M_t, \eta_t, \kappa_t)$ over $F$ for $(F_t, \eta_t, \tilde{\kappa}_t)$.

Write $M_t \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq \prod_{x \in J_t} M_{t,x}$ as a product of fields. As the BT-group $\Sigma$ is acted on by $F_{\mathbb{Q}_p}(\delta) = \prod_{t \in I} F_t \otimes_{\mathbb{Q}} \mathbb{Q}_p$, it is also acted on by $\prod_{x \in J_t} M_{t,x}$ via $M_t \hookrightarrow F_t$, allowing a decomposition up to isogeny

$$\Sigma \simeq \oplus_{t \in I} \oplus_{x \in J_t} \Sigma_{t,x}.$$  

There is induced a $\mathbb{Q}_p$-algebra map $B \otimes_F M_{t,x} \simeq M_0(M_{t,x}) \to \text{End}^0(\Sigma_{t,x})$. Using an idempotent in $M_0(M_{t,x})$, we find a BT-group $\Sigma_{t,x}^{\text{red}}$ such that $\Sigma_{t,x} \simeq (\Sigma_{t,x}^{\text{red}})^{\oplus n}$ with compatible $M_{t,x}$-actions in the isogeny category. We define a rational number

$$n_t := \frac{\text{height}(\Sigma_{t,x}^{\text{red}})}{[M_{t,x} : \mathbb{Q}_p][C^*_t : M_t]^{1/2}}.$$  

Lemma 10.4. The following are true.

(i) The number $n_t$ is an integer and independent of $x$.

(ii) The map

$$(\gamma_0; \gamma, \delta) \mapsto \left( \prod_{t \in I} M_t, (\eta_t), (n_t), (\kappa_t) \right)$$

gives a well-defined map from $KT_b$ to the set of equivalence classes of $p$-adic types over $F$.

(iii) The image of the above map lies in the set described in (ii) of Corollary 8.5.

Remark 10.5. Note that the $p$-adic type $(\prod_{t \in I} M_t, (\eta_t), (n_t), (\kappa_t))$ we constructed above is not necessarily minimal. It may happen that $(M_t, \eta_t, \kappa_t)$ and $(M'_t, \eta'_t, \kappa'_t)$ are equivalent for $t \neq t'$. 
Proof. Assuming we have proved (i), we obtain (ii) as an easy consequence since the construction of the map does not change if we replace \( \gamma_0 \) with a stably conjugate element or \( \delta \) by a conjugate element. To see (iii), observe that

For each slope \( \lambda_k \) of \( \Sigma \) for \( 1 \leq k \leq r \), we have

This verifies (iii) of the lemma.

We begin the proof of (i). As we have \( M_t \hookrightarrow F_t \) and \( B \otimes_F F(\gamma_0) \cong \prod_t B \otimes_F F_t \) acts on \( V \) preserving \( B \)-action and Hermitian pairing, we have a decomposition \( V \cong \bigoplus_t V_t \) where \( V_t \) is a Hermitian \( B \otimes_F M_t \)-module. Let \( r_t := \text{rank}_M V_t \). As \( M_t \otimes \mathbb{Q}_p \cong \prod_{\ell \mid p} M_t, \) we have \( V_t \otimes \mathbb{Q}_p \cong \otimes_{\ell \mid p} V_{t,\ell} \) correspondingly. Then \( V_{t,\ell} \) is a module over \( M_n(M_{t,\ell}) \) which is an \( r_t \)-dimensional vector space over \( M_{t,\ell} \). Therefore \( r_t := r_t/\ell \) is an integer. Since there is an isomorphism \( V(\Sigma_{t,\ell}) \cong V_{t,\ell} \otimes \mathbb{Q}_p \) as \( L \)-vector spaces,

We deal with the case where \( x \mid p \) and \( x \nmid \infty \). Suppose that \( x \) divides a rational prime \( q(\neq p) \).

According to \( M_t \otimes \mathbb{Q}_q \cong \prod_{\ell \mid q} M_{t,\ell} \), we decompose \( V_t \otimes \mathbb{Q}_q \cong \prod_{\ell \mid q} V_{t,\ell} \). Let \( d_{t,\ell} \) be the denominator of \( \text{inv}_{\ell}(B \otimes_F M_{t,\ell}) \), so that \( B \otimes_F M_{t,\ell} \cong M_n/d_{t,\ell}(D_{t,\ell}) \) for some central division algebra \( D_{t,\ell} \) over \( M_{t,\ell} \) with degree \( d_{t,\ell}^2 \). As \( V_{t,\ell} \) is a module over \( B \otimes_F M_{t,\ell} \), it follows that \( n d_{t,\ell} \) divides \( \text{rank}_{M_{t,\ell}} V_{t,\ell} = r_t \). Therefore \( r_t \text{inv}_{\ell}(C_t) = -r_t \text{inv}_{\ell}(B \otimes_F M_{t,\ell}) = 0 \) in \( \mathbb{Z}/\ell \).

A real place \( x \) occurs only when \( F \) and \( M_t \) are totally real fields, for a PEL datum of type (C). We know that \( V_t \otimes \mathbb{R} \) is a Hermitian \( B \otimes_F M_t \otimes \mathbb{R} \)-module. We decompose \( V_t \otimes \mathbb{R} \) into a product of \( B \otimes_F M_{t,\ell} \)-modules \( V_{t,\ell} \) as \( x \) runs over real places of \( M_t \). If \( B \otimes_F M_{t,\ell} \) does not split, \( \text{inv}_{\ell}(C_t) = 0 \). If \( B \otimes_F M_{t,\ell} \) splits, we know that \( r_t \) is even from the existence of a symplectic pairing on \( V_{t,\ell} \).

This is enough for conclusion.

\[ \text{height}(\Sigma_{t,\ell}) = \text{height}(\Sigma_{t,\ell}^\text{red})[B : F]^{1/2} = n_t[M_{t,\ell} : \mathbb{Q}_p][B : F]^{1/2}[C_t : M_t]^{1/2}. \]

For each slope \( \lambda_k \) of \( \Sigma \) for \( 1 \leq k \leq r \) and each place \( \nu \mid p \) of \( F \),

\[ \sum_{t \in I} \sum_{x \in S_{\lambda_k}(M_t)} \text{height}(\Sigma_{t,\ell}) = \sum_{(t,\ell) \text{ with } x | e, \ k(t,\ell) = \lambda_k} \text{height}(\Sigma_{t,\ell}) = \text{height}(\Sigma^k[v^\infty]). \]

Corollary 10.6. The maps in Lemma 10.4 and Corollary 8.5 induce a map from \( KT_b \) to the set of those isomorphism classes \( (A, i) \in AV_{et}^0 \) for which there exists a quasi-isogeny \( j : \Sigma \to A[p^\infty] \) compatible with \( B_{Q_{\ell}} \)-action.

Proof. Immediate from Lemma 10.4 and Corollary 8.5.

As preparation for the definition of the invariants \( \alpha(\gamma_0; \gamma, \delta) \) and \( \beta(\gamma_0; \gamma, \delta) \), we introduce two algebraic groups \( I_0 \) and \( H_0 \) for a Kottwitz triple \( (\gamma_0; \gamma, \delta) \). Using Lemma 10.4, we find an \( F \)-algebra \( M = \prod_t M_t \) appearing in the minimal \( p \)-adic type for \( (\gamma_0; \gamma, \delta) \). Define

\[ H_0(R) = \{ g \in \text{End}_{B \otimes_F M}(V) \otimes_R R | gg^\# \in R^* \} \]
for each $\mathbb{Q}$-algebra $R$. Also define $I_0 := Z_G(\gamma_0)$ so that $I_0(R) = \{g \in \text{End}_{B \otimes F(\gamma_0)}(V) \otimes_{\mathbb{Q}} R | gg^* \in R^* \}$. Clearly $I_0 \subset H_0 \subset G$.

Now we introduce local components $\alpha_v$ and $\beta_v$, which lead to the invariants $\alpha(\gamma_0; \gamma, \delta)$ and $\beta(\gamma_0; \gamma, \delta)$. We begin with $\alpha_v, \beta_v$ for $v \neq p, \infty$. Denoting the $v$-component of $\gamma$ by $\gamma_v \in G(\mathbb{Q}_v)$, there exists $g \in G(\mathbb{Q}_v)$ such that $g \gamma_0 g^{-1} = \gamma_v$. From this, we construct a cocycle $c_v := (\tau \mapsto g^{-1} g^*)$ in $\ker(H^1(Q_v, I_0) \to H^1(Q_v, G))$. In the diagram below which is commutative by Lemma 2.3, we define $\alpha_v \in A_v(I_0)$ and $\beta_v \in A_v(H_0)$ as the image of $c_v$. Then $\alpha_v$ and $\beta_v$ map to the trivial element in $A_v(G)$. We will also view $\alpha_v, \beta_v$ as elements of $X^*(Z(\hat{I}_0)^{(p)})$ and $X^*(Z(H_0)^{(p)})$, respectively.

\[
\begin{array}{ccc}
H^1(Q_v, I_0) & \longrightarrow & H^1(Q_v, H_0) \quad \longrightarrow \quad H^1(Q_v, G) \\
A_v(I_0) & \longrightarrow & A_v(H_0) \quad \longrightarrow \quad A_v(G)
\end{array}
\] (28)

Next we deal with $\alpha_p$ and $\beta_p$. We freely borrow notation from §4. Observe that the $L$-vector spaces $V \otimes_{\mathbb{Q}} L$ and $V(\Sigma, \lambda_\Sigma, i_\Sigma)$ are Hermitian modules with respect to the natural action of $B \otimes_F F_{Q_p}(\gamma_0) \simeq B \otimes_F F_{Q_p}(\delta)$, where the last isomorphism is chosen so that $\gamma_0 \to \delta$. As $V \otimes_{\mathbb{Q}} L$ is equivalent to $V(\Sigma, \lambda_\Sigma, i_\Sigma)$ as Hermitian $B \otimes F_{Q_p}(\gamma_0) \otimes_{\mathbb{Q}} L$-modules (by Steinberg’s vanishing theorem of $H^1$), choose any such equivalence $f$ and transport the map $\Phi$ on $V(\Sigma, \lambda_\Sigma, i_\Sigma)$ to $\Phi_f$ on $V \otimes L$. Define $\tilde{b}_b$ by the relation $\Phi_0 = \tilde{b}_b(1 \otimes \sigma)$. Then $\tilde{b}_b$ belongs to $I_0(L)$ and defines $b_0 \in B((I_0)_{Q_p})$, which is independent of the choice of the above isomorphisms. Define $\alpha_p := \kappa_{I_0}(b_0)$. (In view of Remark 4.15, we may replace $V(\Sigma, \lambda_\Sigma, i_\Sigma)$ by $\mathcal{V}(\tilde{b})$ in the definition of $\alpha_p$. The action of $\delta$ on $V(\tilde{b})$ is defined via the $Q_p$-isomorphism $J_b(\mathbb{Q}_p) \simeq \text{Aut}(\mathcal{V}(\tilde{b}))$, which is canonical up to $J_b(\mathbb{Q}_p)$-conjugacy.)

Consider the following commutative diagram coming from the functoriality of the map $\kappa_v$ (see §4).

The element $\beta_p \in X^*(Z(H_0)^{(p)})$ is defined to be the image of $\alpha_p$. Since $b_0$ maps to $b$ in $B(G_{Q_p}, -\mu)$, both $\alpha_p$ and $\beta_p$ map to $-\mu_1 \in X^*(Z(\hat{G})^{(p)})$ by the bottom arrows. Recall that $\kappa_{Q_0}(b) = -\mu_1$.

Finally we describe $\alpha_\infty, \beta_\infty$. We can choose an elliptic maximal real torus $T$ of $G$ containing $\gamma_0$. We also choose a $G(\mathbb{R})$-conjugate of $h : \text{Res}_{\mathbb{C}/\mathbb{R}}(G_m) \to G$ factoring through $T$ and use it to define $\mu_h$ (see §5). Then we see that $\mu_h$ belongs to $X_*(T) = X^*(\hat{T})$ and that the image of $\mu_h$ in $X^*(\hat{T})^{(\infty)}$ is independent of choices (See [Kot90, p.167]). By restricting this image via the canonical embedding $Z(\hat{I}_0) \hookrightarrow Z(\hat{H}_0) \hookrightarrow \hat{T}$, we get elements

\[
\alpha_\infty \in X^*(Z(\hat{I}_0)^{(\infty)}) \quad \text{and} \quad \beta_\infty \in X^*(Z(\hat{H}_0)^{(\infty)}).
\]

We are ready to define the elements $\alpha(\gamma_0; \gamma, \delta) \in \mathfrak{R}(I_0/\mathbb{Q})^D$ and $\beta(\gamma_0; \gamma, \delta) \in \mathfrak{R}(H_0/\mathbb{Q})^D$. As we assumed that $\gamma_0$ is elliptic, it follows that $\mathfrak{R}(I_0/\mathbb{Q}) = (\bigcap_v Z(\hat{I}_0)^{(v)} Z(\hat{G}))/Z(\hat{G})$ ([Kot90, p.166]). We extend $\alpha_v$ to an element $\alpha'_v$ of $X^*(Z(\hat{I}_0)^{(v)} Z(\hat{G}))$ so that on $Z(\hat{G})$, $\alpha'_v$ is $-\mu_1$ if $v = p$, and $\mu_1$ if $v = \infty$, and trivial if $v \neq p, \infty$. Similarly, we define $\beta'_v \in X^*(Z(\hat{H}_0)^{(v)} Z(\hat{G}))$. The elements
\( \alpha(\gamma_0; \gamma, \delta) \) in \( \mathfrak{H}(I_0/\mathbb{Q})^D \) and \( \beta(\gamma_0; \gamma, \delta) \) in \( \mathfrak{H}(H_0/\mathbb{Q})^D \) are defined by

\[
\alpha(\gamma_0; \gamma, \delta) = \left( \prod_{v \notin p, \infty} \alpha_v^I|_{\mathfrak{H}(I_0/\mathbb{Q})} \right) \cdot (\alpha^\prime_0 \alpha^\prime_\infty)|_{\mathfrak{H}(I_0/\mathbb{Q})},
\]

\[
\beta(\gamma_0; \gamma, \delta) = \left( \prod_{v \notin p, \infty} \beta_v^I|_{\mathfrak{H}(H_0/\mathbb{Q})} \right) \cdot (\beta^\prime_0 \beta^\prime_\infty)|_{\mathfrak{H}(H_0/\mathbb{Q})}.
\]

In terms of the vanishing of \( \alpha(\gamma_0; \gamma, \delta) \), we want to single out those Kottwitz triples which are expected to come from the moduli data of Igusa varieties. This motivates the following definition.

**Definition 10.7.** A Kottwitz triple \( (\gamma_0; \gamma, \delta) \) is called effective if \( \alpha(\gamma_0; \gamma, \delta) \) is trivial. We define the set \( KT_{b,0}^H \) to be the subset of \( KT_b \) consisting of the equivalence classes of effective Kottwitz triples.

For later use, we record two lemmas regarding a Kottwitz triple \( (\gamma_0; \gamma, \delta) \). Define \( I_\delta := Z_I(\delta) \).

**Lemma 10.8.** The group \( I_\delta \) is an inner form of \( I_0 \) over \( \mathbb{Q}_p \).

**Proof.** Let \( s \) be the positive integer in the decency equation (6) for \( \bar{b} \). Denote by \( \sigma \) the Frobenius element in \( \text{Gal}(L/\mathbb{Q}_p) \) or \( \text{Gal}(L_s/\mathbb{Q}_p) \). To avoid confusion, for any \( \tau \in \Gamma(p) \) we will sometimes write \( \tau_M \) (resp. \( \tau_J \)) for the \( \tau \)-action on the points of \( M_b \) (resp. \( J_b \)). Recall from Lemma 4.2 that there is an isomorphism \( \psi : M_b \xrightarrow{\sim} J_b \) over \( L_s \) such that \( \psi^{-1} \psi^\sigma = \text{Int}(\bar{b}) \). Viewing \( \psi \) as a \( \mathbb{Q}_p \)-morphism by base change, define \( c_{\tau} \in M_b(\overline{\mathbb{Q}}_p) \) for each \( \tau \in \Gamma(p) \) so that \( \text{Int}(c_{\tau}) = \psi^{-1} \psi^\tau \). (Here \( \psi^\tau \) is \( \tau_M \psi^{-1} \) by definition.) The condition \( \delta \sim_{st} \gamma_0 \) means that there exists \( x \in G(\overline{\mathbb{Q}}_p) \) such that

\[
\psi^{-1}(\delta) = x \gamma_0 x^{-1}.
\]

Define a \( \overline{\mathbb{Q}}_p \)-morphism \( \theta := \psi \circ \text{Int}(x) \). This induces a \( \overline{\mathbb{Q}}_p \)-isomorphism \( I_0 \xrightarrow{\sim} I_\delta \). Indeed, \( \theta(\gamma_0) = \delta \) implies that \( \theta \) induces a \( \overline{\mathbb{Q}}_p \)-isomorphism \( Z_{M_b}(\gamma_0) \xrightarrow{\sim} I_\delta \), but \( I_0 = Z_{M_b}(\gamma_0) \) by the acceptability of \( \delta \). For any \( \tau \in \Gamma(p) \),

\[
\theta^{-1} \theta^\tau = \text{Int}(x^{-1}) \psi^{-1} \psi^\tau \text{Int}(x^{\tau_M}) = \text{Int}(x^{-1} c_{\tau} x^{\tau_M}).
\]

On the other hand,

\[
x^{\tau_M} \gamma_0 x^{-\tau_M} = \psi^{-1}(\delta)^{\tau_M} = \text{Int}(c_{\tau}) \psi^{-1}(\delta^{\tau_M}) = \text{Int}(c_{\tau}) \psi^{-1}(\delta) \tag{30}
\]

By (29) and (30), we conclude that \( x^{-1} c_{\tau} x^{\tau_M} \in I_0(\overline{\mathbb{Q}}_p) \). Therefore \( I_\delta \) is a \( \mathbb{Q}_p \)-inner form of \( I_0 \) given by \( \tau \mapsto \text{Int}(x^{-1} c_{\tau} x^{\tau_M}) \) in \( H^1(\mathbb{Q}_p, \text{Int}(I_0)) \).

\( \square \)

**Lemma 10.9.**

\[
\alpha_p(\gamma_0; \gamma, \delta') = \alpha_p(\gamma_0; \delta) + \text{inv}_p(\delta, \delta') \tag{31}
\]

\[
\alpha_p(\gamma_0'; \delta) = \alpha_p(\gamma_0; \delta) + \text{inv}_p(\gamma_0, \gamma_0') \tag{32}
\]

hold in \( X^*(Z(I_0)^{\Gamma(p)}) \), where \( \text{inv}_p(\delta, \delta') \) is viewed as an element of \( X^*(Z(I_0)^{\Gamma(p)}) \) via the canonical \( \Gamma(p) \)-equivariant isomorphism \( Z(I_0) \simeq Z(I_0) \) (by Lemma 10.8).
Proof. Before the proof, we record a general fact in the first paragraph. Let $r$ be a positive integer. Let $\gamma_1$ and $\gamma_2$ be semisimple elements of $G(L_v)$ which are conjugate in $G(L)$. Then $\gamma_1$ and $\gamma_2$ are conjugate in $G(L)$. This follows from an easy application of Steinberg’s vanishing theorem. Now suppose that $\gamma_1, \gamma_2 \in G(Q_p)$. Let $x_0 \in G(Q_p)$ and $x \in G(L)$ be such that $\gamma_2 = x_0 \gamma_1 x_0^{-1}$ and $\gamma_2 = x \gamma_1 x^{-1}$. Let $I_1 := Z_G(\gamma_1)$. Recall ([Kot85, 1.7-1.8]) that the map $H^1(Q_p, I_1) \to B(I_1)$ of (5) is given by

$$H^1(Q_p, I_1) \hookrightarrow H^1(T/Q_p, I_1) \cong H^1(L/Q_p, I_1) \cong B(I_1)$$

(33)

where the inverse of the second map is given by the inflation map. (It is an isomorphism by Steinberg’s vanishing theorem.) The last isomorphism sends a cocycle $\sigma \mapsto \gamma$ to the image of $\gamma$ in $B(I_1)$. It is easy to see that the cocycle $\tau \mapsto x_0^{-1} x_1^\sigma$ (up to coboundary) in $H^1(Q_p, I_1)$ maps to the image of $x^{-1} x^\sigma$ in $B(I_1)$ under the composite map of (33).

Now we begin to prove (31). Recall from the proof of Lemma 10.8 that we have an $L_v$-isomorphism $\psi : \mathcal{M}_b \cong J_b$. Let $y \in G(L)$ be an element such that $\psi^{-1}(\delta) = y \gamma_0 y^{-1}$. Such a $y$ exists by the discussion of the last paragraph since $\psi^{-1}(\delta)$ and $\gamma_0$ are conjugate in $G(Q_p)$. We may view $y$ as an equivalence (in the sense of §2) from $V \otimes Q L = V_{Q_p} \otimes Q_v L$ with the natural Hermitian $B \otimes_F F_{Q_p}(\gamma_0) \otimes Q_v L$-module structure onto $V \otimes Q_L$ with the natural Hermitian $B \otimes_F F_{Q_v}(\delta) \otimes Q_v L$-module structure, where the latter is the underlying Hermitian structure of the $G$-isocrystal $\mathbb{V}(\hat{b})$ (defined in Remark 4.15). Recall that $V(\Sigma, \lambda_0, \lambda_z)$ may be replaced by $\mathbb{V}(\hat{b})$ in the definition of the $\alpha_{p}$-invariant. We let $y$ play the role of $f$ in the definition of $\alpha_{p}(\gamma_0; \delta)$. Then

$$\bar{b}_y = y^{-1} b y^\sigma$$

and $\alpha_{p}(\gamma_0; \delta) = \kappa_{\text{Is}}(b_y)$. On the other hand, it is easy to check that $\psi \circ \text{Int}(y)$ gives an $L$-isomorphism $I_0 \cong I_\delta$, which will be called $\theta$.

Consider the following commutative diagram where the left rectangle comes from (5) and the right rectangle from [Kot97, 4.13].

$$
\begin{array}{ccc}
H^1(Q_p, I_\delta) & \xrightarrow{\kappa_{I_\delta}} & B(I_\delta) \\
\downarrow{\kappa_{\text{Is}, p}} & \quad & \downarrow{\kappa_{I_\delta}} \\
A_p(I_\delta) & \xrightarrow{\kappa_{I_\delta}} & X^*(Z(I_\delta)^\Gamma(p)) \\
\end{array}
$$

The right top horizontal arrow is induced by $i \mapsto \theta^{-1}(i) \bar{b}_y$. The right bottom arrow is simply the addition by $\kappa_{\text{Is}}(b_y)$ via the canonical identification $Z(I_\delta) = Z(I_0)$. On the other hand, there exist $j_0 \in J_0(Q_p)$ and $j \in J_0(L)$ such that $\delta' = j_0 \delta j_0^{-1}$ and $\delta' = j \delta j^{-1}$. Let $c(\delta, \delta') \in H^1(Q_p, I_\delta)$ be given by the cocycle $\tau \mapsto j_0^{-1} j_0 \bar{b}_y$. By the earlier discussion $c(\delta, \delta')$ maps to $j^{-1} j_0 \bar{b}_y$ under $H^1(Q_p, I_\delta) \to B(I_\delta)$. Observe that $\bar{b}_y$ may be defined using the equivalence $\psi(j) y$ in the same way as $b_y$ was defined using $y$. Thus

$$\bar{b}_y = (jy)^{-1} b (jy)^{\sigma} = y^{-1} \psi(j)^{-1} b \psi(j)^{\sigma} y^{\sigma}.$$

The image of $c(\delta, \delta')$ in $B(I_\delta)$ is given by $\theta^{-1}(j^{-1} j_0 \bar{b}_y \in I_0(L)$, which is equal to

$$y^{-1} \psi^{-1}(j^{-1} j_0) y^\sigma - \bar{b}_y = y^{-1} \psi^{-1}(j^{-1}) \text{Int}(\bar{b})(\psi^{-1}(j)^{\sigma} \bar{b} y^{\sigma} = y^{-1} \psi^{-1}(j^{-1}) \bar{b} \psi^{-1}(j)^{\sigma} y^{\sigma} = \bar{b}_y.$$
which is nothing but (31).

We prove (32) in a similar way using an analogue of the diagram (34), with $I_\delta$ (resp. $I_0$) replaced by $I_0$ (resp. $I_0'$) where $I_0' := Z_G(\gamma_0)$.

11 Auxiliary invariants and vanishing of invariants

Throughout this section, let $(A, i)$ be an object of $AV^2_B$ such that there is a quasi-isogeny $j : \Sigma \to A[p^\infty]$ compatible with $Q_B \otimes_\mathbb{Z} Z_{p'}$-action. Let $M := Z(\text{End}_{\mathbb{Q}_p}^0(A))$. Let $\lambda$ be a $(B \otimes_F M, \star \otimes c)$-polarization of $A$. We do not suppose that $(A, \lambda, i)$ represents an equivalence class of $\text{PIC}_B$. Let $(\gamma_0; \gamma, \delta) \in KT_B$ be such that it corresponds to the $p$-adic type for $(A, i)$ via Lemma 10.4. Recall that an $F$-embedding $M \hookrightarrow F(\gamma_0)$ is given. Let $i' : B \otimes_F F(\gamma_0) \hookrightarrow \text{End}_{\mathbb{Q}_p}^0(A)$ be any mapping extending $i$.

Let us define $\alpha(\gamma_0; (A, \lambda, i'))$ and $\beta(\gamma_0; (A, \lambda, i))$. First define $\alpha_v$ and $\beta_v$ for $v \neq p, \infty$. Let $(c_v)_{v \neq p, \infty}$ be the element in $H^1(Q, I_0(\mathbb{K}_{\infty}^p))$ measuring the difference of $V \otimes \mathbb{K}_{\infty}^p$ and $(V^p A)_\lambda$ as Hermitian $B \otimes_F F(\gamma_0) \otimes_Q \mathbb{K}_{\infty}^p$-modules, using Lemma 3.3. (The Hermitian $B \otimes_F F(\gamma_0) \otimes_Q \mathbb{K}_{\infty}^p$-module structure on $V \otimes \mathbb{K}_{\infty}^p$ is induced by $i'$.) Let $(d_v)_{v \neq p, \infty}$ be the element in $H^1(Q, H_0(\mathbb{K}_{\infty}^p))$ measuring the difference of $V \otimes \mathbb{K}_{\infty}^p$ and $V^p A$ as Hermitian $B \otimes_F F(\gamma_0) \otimes_Q \mathbb{K}_{\infty}^p$-modules. Using the maps in the diagram (28), we get elements $\alpha_v \in A_v(I_0)$ and $\beta_v \in A_v(H_0)$ corresponding to $c_v$ and $d_v$. We will also view $\alpha_v$ and $\beta_v$ as elements of $X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$ and $X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$, respectively. To define $\alpha_p$ for $(\gamma_0; (A, \lambda, i'))$ and $\beta_p$ for $(\gamma_0; (A, \lambda, i))$, we only need to replace $V(\Sigma, \lambda_\Sigma; i_\Sigma)$ equipped with $B \otimes F_{Q, \gamma}(\delta)$-action by $V(A[p^\infty], \lambda, i')$ with $B \otimes F_{Q, \gamma}(\gamma_0)$-action in the definition of $\alpha_p$ and $\beta_p$ for $(\gamma_0; \gamma, \delta)$. The resulting elements $\alpha_p$ in $X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$ and $\beta_p$ in $X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$ map to $-\mu_1 \in X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$ as before. The local components are defined to be the same as for $(\gamma_0; \gamma, \delta)$. Finally the element $\alpha(\gamma_0; (A, \lambda, i'))$ in $X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$ is defined to be $\prod_v \alpha_v|_{\mathbb{K}(\mathbb{I}_0)^{\Gamma}}$. Likewise, $\beta(\gamma_0; (A, \lambda, i))$ in $X^*(\mathbb{K}_{\infty}^p)^{\Gamma(\gamma_0)}$ is defined to be $\prod_v \beta_v|_{\mathbb{K}(\mathbb{I}_0)^{\Gamma}}$.

Now we introduce a variant of the above construction. Here we assume that $(\gamma_0; (A, \lambda, i), [a]) \in FP^AV_B$. Recall that there exists an embedding $\iota_{\gamma_0; (A, \lambda, i), [a]} : H_{\gamma_0; (A, \lambda, i), [a]}(\mathbb{K}_{\infty}^p) \hookrightarrow G(\mathbb{K}_{\infty}^p) \times J_0(Q_p)$. Let $\gamma_0 \in G(\mathbb{Q})$ be as before. (Namely, there exist $\gamma$ and $\delta$ such that $(\gamma_0; \gamma, \delta)$ lies in $KT_B$ and corresponds to $(\gamma, \delta)$.) Suppose that $\iota_{\gamma_0; (A, \lambda, i), [a]}(a)$ is conjugate to $\gamma_0$ in $G(\mathbb{K}_{\infty}^p)$, via any $J_0(Q_p) \hookrightarrow G(Q)$ as in Definition 10.1. Then we can define $\alpha(\gamma_0; (A, \lambda, i), [a])$ in $X^*(\mathbb{K}_{\infty}^p)^{F(\gamma_0)} = \prod_v \alpha_v|_{\mathbb{K}(\mathbb{I}_0)^{\Gamma}}$ where $\alpha_v$ are as follows. In case $v \neq p, \infty$, we reuse the previous definition of $\alpha_v$ for $\alpha(\gamma_0; (A, \lambda, i'))$ where $(V^p A)_\lambda$ is viewed as a Hermitian $B \otimes F(\gamma_0) \otimes_Q \mathbb{K}_{\infty}^p$ via an isomorphism $F(a) \otimes_Q \mathbb{K}_{\infty}^p \simeq F(\gamma_0) \otimes_Q \mathbb{K}_{\infty}^p$ such that $a \mapsto \gamma_0$. For $v = p$ and $v = \infty$, the definition of $\alpha_v$ is the same as in the case of $\alpha(\gamma_0; (A, \lambda, i'))$.

We will need yet another auxiliary invariant where no reference to $(\gamma_0; \gamma, \delta)$ is made. Let $(A, i)$ be as in the beginning of this section. (Drop the assumption $((A, \lambda, i), [a]) \in FP^AV_B$.) Suppose that $N$ is a product of fields which are totally real or CM, and that $N$ embeds into $\text{End}_{\mathbb{Q}_p}^0(A)$ as a maximal commutative semisimple $F$-subalgebra. Then $i$ naturally extends to $i' : B \otimes_F N \to \text{End}^0(A)$. Let $\lambda$ be a $(B \otimes_F N, \star \otimes c)$-polarization of $A$ with respect to $i'$. Assume that $W$ is a $B \otimes_F N$-module with a $\star \otimes c$-Hermitian pairing $\langle \cdot, \cdot \rangle_W$ such that $W \simeq V$ as $B$-modules. Define a $\mathbb{Q}$-torus $T$ by $T(R) = \{ g \in N \otimes_{\mathbb{Q}} R \mid gg^* \in R^+ \}$ for any $\mathbb{Q}$-algebra $R$. Then we construct

$$\alpha(N, W; (A, \lambda, i)) = \prod_v \alpha_v|_{F^v} \in X^*(\mathbb{T}_W)$$

where the local components $\alpha_v \in X^*(\mathbb{T}_W(v))$ are given as follows. The elements $\alpha_v$ for $v \neq \infty$ are defined in the same way as for $\alpha(\gamma_0; (A, \lambda, i'))$ except that we replace $V$, $I_0$, $F(\gamma_0)$ respectively
with $W$, $T$, $N$ in the previous definition. To describe $\alpha_{\infty}$, choose an $\mathbb{R}$-algebra homomorphism $h' : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(W_2)$ such that the pairing $(v, w) \mapsto \langle v, h'(\sqrt{-1}w)w \rangle_W$ is positive definite on $W_2$. Viewing $h'$ as an $\mathbb{R}$-morphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G_m \to G_{\mathbb{R}}$, choose a $G(\mathbb{R})$-conjugate $h''$ of $h'$ that factors through $T$. We get $\mu_{n''} \in X_*(T) = X^*(T)$ from $h''$ (as in §5). Via restriction this gives the element $\alpha_{\infty} \in X^*(\hat{T}^{(\infty)})$, which is independent of the choice of $h'$ and $h''$.

We conclude this section with the following lemma which is an important step in proving the vanishing of Kottwitz invariants in certain cases.

**Lemma 11.1.** Suppose that $(A, i)$ is an object of $\mathcal{A}^{0}_B$ such that there is an isomorphism $j : \Sigma \cong A[p^\infty]$ compatible with $O_B \otimes Z_{\mathbb{Z}}$-action. Put $M := Z(\mathcal{A}^{0}(A, i))$. Suppose that $N$ is a product of fields which are totally real or CM, and that $N$ embeds into $\mathcal{A}(A, i)$ as a maximal commutative semisimple $F$-subalgebra. Then we can find a $(B \otimes F, \ast \otimes c)$-polarization $\lambda_0 : A \to A'V$ and a $B \otimes F$-module $W_0$ with a $\ast \otimes c$-Hermitian pairing $(\cdot, \cdot)_0 : W_0 \times W_0 \to Q$ such that

(i) $W_0 \otimes Q A^{\infty,p}$ and $(V^p_0 A)_{\lambda_0}$ are equivalent as $B \otimes Q A^{\infty,p}$-Hermitian modules where $(V^p_0 A)_{\lambda_0}$ is the $B \otimes F N \otimes Q A^{\infty,p}$-module $V^pA$ with the $\lambda_0$-Weil pairing, and

(ii) $W_0 \otimes Q R$ and $V \otimes Q R$ are equivalent as $B \otimes Q R$-modules with Hermitian pairings $(\cdot, \cdot)_0$ and $(\cdot, \cdot)$ (the latter from the PEL datum), respectively.

**Remark 11.2.** Given any object $(A, i)$ of $\mathcal{A}^{0}_B$ such that there exists a quasi-isogeny $j : \Sigma \cong A[p^\infty]$ compatible with $O_B \otimes Z_{\mathbb{Z}}$-action, we can always find an object $(A', i')$ which is isomorphic in $\mathcal{A}^{0}_B$ to $(A, i)$ such that there is an isomorphism $j : \Sigma \cong A[p^\infty]$ compatible with $O_B \otimes Z_{\mathbb{Z}}$-action. This is possible using the argument in the proof of Lemma 7.1.

**Proof.** Note that $i$ naturally extends to an $F$-algebra map $B \otimes F N \hookrightarrow \mathcal{A}^{0}(A)$, which we call $i'$. Write $N = \prod_i N_i$ as a product of fields, and decompose $(A, i') = \oplus_{i \in i}(A_i, i_i)$ accordingly where $i_i$ is a $\mathbb{Q}$-algebra map $B \otimes F N_i \hookrightarrow \mathcal{A}^{0}(A_i)$. Then $N_i$ is a maximal commutative subalgebra of $\mathcal{A}^{0}(A_i, i_i)$ for each $t$. By maximality of $N_t$, there is an isomorphism $B \otimes F N_t \cong M(A_t)$ by Lemma 9.2. We may choose a $(B \otimes F N_t, \ast \otimes c)$-polarization $\lambda_t : A_t \to A'_t$ for each $t$. By putting them together, we have a $(B \otimes F N, \ast \otimes c)$-polarization $\lambda_0 : A \to A'V$.

Arguing as in [HT01, p.170-171], we find a lifting $(\tilde{A}_t, \tilde{\lambda}_t, \tilde{\gamma}_t)$ of $(A_t, \lambda_t, i_t)$ with respect to the fixed reduction map $\tau_p : O_F^p \to \mathbb{F}_p$, where $\tilde{A}_t$ is an abelian scheme over $O_{F^p}^{\ast}$, $\tilde{\lambda}_t$ is a polarization of $\tilde{A}_t$, and $\tilde{\gamma}_t : B \otimes F N_t \to \mathcal{A}^{0}(A_t)$ such that $\tilde{\gamma}_t$ restricts to $\ast \otimes c$ via $\tilde{\gamma}_t$.

Set $W_t := H_1(\tilde{A}_t \times O_{F^p}^{\ast}, (\mathbb{C}, Q))$. This is a $B \otimes F N_t$-module with a $\ast \otimes c$-Hermitian pairing coming from $\tilde{\lambda}_t$. So $W_0 := \oplus_{t \in T} W_t$ is equipped with a $B \otimes F N$-module structure with a $\ast \otimes c$-Hermitian pairing $(\cdot, \cdot)_0 : W_0 \times W_0 \to Q$. By construction we see that $W_0 \otimes A^{\infty,p}$ is equivalent to $V^pA$ as Hermitian $B \otimes F N \otimes Q A^{\infty,p}$-modules.

It remains to prove that $W_0 \otimes R \cong V \otimes R$ as Hermitian $B \otimes R$-modules. Put $\hat{A} = \prod_i \hat{A}_i$. Observe that $\hat{A}$ is a module over $O_F \otimes Z_{p^\infty} \cong \prod_i O_{F^p}^{\ast}$ where $\xi$ runs over the set $\text{Hom}_{Z}(O_F, O_{F^p}^{\ast})$. Accordingly we decompose $\hat{A} = \oplus_{\xi}(\hat{A})_{\xi}$. Similarly, we have decompositions of $O_F \otimes Z_{p^\infty}$-modules $\hat{A} = \oplus_{\xi}(\hat{A})_{\xi}$ and $\hat{\Sigma} = \oplus_{\xi}(\hat{\Sigma})_{\xi}$ where $\xi$ runs over the set $\text{Hom}_{Z}(O_F, \mathbb{F}_p)$. We have

$$\text{rank}_{O_{F^p}^{\ast}}(\hat{A})_{\xi} = \dim_{\mathbb{F}_p} \langle \text{dim}_{\mathbb{F}_p} (\hat{A})_{\xi} \rangle_{\tau_p, \xi} = \dim_{\mathbb{F}_p} (\hat{\Sigma})_{\xi}$$

in which the last equality follows from the determinant condition imposed on $\Sigma$ (see §5).

Now observe that there is an isomorphism of $O_F \otimes \mathbb{C}$-modules

$$W_0 \otimes \mathbb{Q} \cong (\hat{\Sigma})_{\xi} \otimes_{O_{F^p}^{\ast}, \xi} \mathbb{C}.$$
Lemma 12.1. Let $(\cdot , \gamma, \delta)$ be the image of $(\gamma, \delta)$ in $\text{Hom}(\mathcal{O}_\tau \otimes \mathbb{C}, \mathbb{C})$, we decompose the left hand side as $W_0 \otimes \mathbb{C} \simeq \bigoplus \tau(W_0 \otimes \mathbb{C})$. We deduce from (35) that

$$\dim_{\mathbb{C}}(W_0 \otimes \mathbb{C})_\tau = \dim_{\mathbb{C}} B_1.$$  

Hence $W_0 \otimes \mathbb{R}$ and $V \otimes \mathbb{R}$ are isomorphic $B \otimes \mathbb{R}$-modules. In the case of a PEL datum of type $(C)$ (i.e. symplectic case), we are done since there is a unique Hermitian $B \otimes \mathbb{R}$-module structure on $V \otimes \mathbb{R}$ up to equivalence. In the case of type $(A)$, the argument of [HT01, p.172-173] shows that $W_0 \otimes \mathbb{R}$ with $(\cdot , \gamma, \delta)$ is equivalent to $V \otimes \mathbb{R}$ with $(\cdot , \gamma, \delta)$ as Hermitian $B \otimes \mathbb{R}$-modules. (The argument of Harris and Taylor shows that the classifying invariants ([HT01, p.49]) for the two Hermitian pairings are respectively given by the numbers $\dim_{\mathbb{C}}(W_0 \otimes \mathbb{C})_\tau$ and $\dim_{\mathbb{C}} B_1$, which are the same by the above identity.)

Corollary 11.3. Suppose that $N$, $W_0$ and $(A, \lambda_0, i)$ are as in Lemma 11.1. Then $\alpha(N, W_0; (A, \lambda_0, i))$ is trivial.

Proof. By Lemma 11.1, $\alpha_v(N, W_0; (A, \lambda_0, i))$ vanishes when $v \neq p, \infty$. The key fact that $\alpha_p \alpha_\infty$ is trivial is proved using the same argument as in the last paragraph of §13 in [Kot92].

12 Main lemmas

First we will describe how we associate a Kottwitz triple $(\gamma_0, \gamma, \delta)$ to an element $((A, \lambda, i), [a]) \in F \mathcal{P}_b^{AV}$. This will lead to a natural map from the set $F \mathcal{P}_b^{AV}$ to the set $KT_b$.

We begin with a fixed element $((A, \lambda, i), [a]) \in F \mathcal{P}_b^{AV}$ and put $z = [(A, \lambda, i)]$. Recall that we defined an embedding $\iota_z : H_2(A^{\infty}) \hookrightarrow G(A^{\infty,p}) \times J_0(\mathbb{Q}_p)$ which is well-defined up to $G(A^{\infty,p}) \times J_0(\mathbb{Q}_p)$-conjugacy. We simply let $(\gamma, \delta)$ be the image of $a \in H_2(A^{\infty})$ under $\iota_z$.

It requires more effort to determine the element $\gamma_0$. The point is that there is an $F$-algebra embedding $i' : F(a) \hookrightarrow \text{End}_B(V)$ compatible with involutions $c$ and $\#$, respectively on the source and the target, by Lemma 14.1 of [Kot92]. In fact, we need to check two implicit assumptions underlying Lemma 14.1 of Kottwitz since it is those assumptions that make his proof work. Firstly, we verify that there exists an $F$-algebra embedding of $F(a)$ into $\text{End}_B(V)$ (with no condition on involutions). This can be checked locally at every place $v$. For $v \neq p, \infty$, $\iota_z$ gives such an embedding. For $v = p$, it is enough to remark that $B_{\mathbb{Q}_p}$ splits. For $v = \infty$, one can easily check case by case for types $(A)$ and $(C)$. Secondly, we verify that the $\mathbb{Q}_p$-group $G_0$ given by $G_0(\mathbb{Q}_p) = \{ g \in \text{End}_B(V)_{\mathbb{Q}_p} | gg^\# = 1 \}$ is quasi-split over $\mathbb{Q}_p$. This follows from our original assumption on the PEL datum.

Now we are ready to describe $\gamma_0$. Using the embedding $i'$ in the last paragraph, we set $\gamma_0$ to be the element $i'(a)$ in $\text{End}_B(V)$. As $i'$ is compatible with involutions, we see that $\gamma_0$ lies in $G(\mathbb{Q})$. Note that by construction there is an isomorphism of $F$-algebras $F(\gamma_0) \simeq F(a)$ such that $\gamma_0 \mapsto a$.

It remains to show that the triple $(\gamma_0; \gamma, \delta)$ we just constructed is a Kottwitz triple and well-defined up to equivalence. The well-definedness is immediate from the construction: the triple changes only within its equivalence class as we vary the choice of $i$ and $\iota_z$ and the representative $((A, \lambda, i), [a])$ in its equivalence class in $F \mathcal{P}_b^{AV}$. The element $\gamma_0$ is semisimple and elliptic over $\mathbb{R}$ since $a$ is so. The acceptability of $\delta$ is inherited from $a$. By construction $\gamma_0$ and $\gamma$ are conjugate in $G(A^{\infty,p})$ and $\gamma_0$ and $\delta$ are conjugate in $G(\overline{\mathbb{Q}}_p)$.

Lemma 12.1. The image of the natural map from $F \mathcal{P}_b^{AV}$ to $KT_b$ defined above is contained in $KT_b^{\text{eff}}$.

Proof. Let $(\gamma_0; \gamma, \delta)$ be the image of $((A, \lambda, i), [a])$. The long exact sequence arising from

$$1 \to Z(\widehat{G}) \to Z(\widehat{I}_0) \to Z(\widehat{I}_0)/Z(\widehat{G}) \to 1$$

37
yields a natural map $Z(\hat{I}_0)^\Gamma \to \mathfrak{R}(I_0/\mathbb{Q})$ which is surjective ([Kot92, p.425]). Thus we get an injection of groups $\mathfrak{R}(I_0/\mathbb{Q})^D \hookrightarrow X^*(Z(\hat{I}_0)^\Gamma)$. The Kottwitz invariants are defined so that $\alpha(\gamma_0; \gamma, \delta)$ maps to $\alpha(\gamma_0; (A, \lambda, i), [a])$ under this map. Therefore the proof of the lemma boils down to showing that $\alpha(\gamma_0; (A, \lambda, i), [a])$ is trivial.

Before further reduction steps, we need some preparation. Set $I = Z_{H_0}(a)$. Arguing as in the top of [Kot92, p.424], we see that the $\mathbb{Q}$-groups $I_0$ and $I$ are inner forms of each other. Note that there are natural inclusions $I_0(\mathbb{Q}) \subset \text{End}_{\mathfrak{R}F} F(\gamma_0)(V)$ and $I(\mathbb{Q}) \subset \text{End}_{\mathfrak{R}F} F(\gamma_0)(A)$. We choose a maximal torus $T$ of $I$ so that $T$ is elliptic at $\infty$ and all the finite places where $I_0$ is not quasi-split. Then $T$ transfers to $I_0$ locally at every place. As $T$ is elliptic over $\mathbb{R}$, the argument at the end of proof of Lemma 14.1 in [Kot92] shows that $T$ transfers globally to $I_0$. Let $N$ be the centralizer of $T(\mathbb{Q})$ in $\text{End}_{\mathfrak{R}F} F(\gamma_0)(A)$. Then $N$ is a product of fields which are CM or totally real. Moreover $N$ is a maximal commutative semisimple $F$-subalgebra of $\text{End}_{\mathfrak{R}F} F(\gamma_0)(A)$ equipped with an $F$-algebra embedding $F(a) \hookrightarrow N$. By the second paragraph of [Kot92, p.426], the transfer of $T$ into $I_0$ provides an inclusion $N \hookrightarrow \text{End}_{\mathfrak{R}F} F(\gamma_0)(V)$ which maps $a \in N$ to $\gamma_0$. To summarize we have the following commutative diagram where all maps are compatible with involutions (complex conjugation $c$ on $F(\gamma_0)$, $F(a)$, $N$; $\#$ on $\text{End}_B(V)$; $\dagger\lambda$ on $\text{End}_{\mathfrak{R}F} F(\gamma_0)(A)$).

\[
\begin{array}{ccc}
F(\gamma_0) & \sim \rightarrow & F(a) \\
\gamma_0 = a & & \\
\text{End}_B(V) & & \text{End}_{\mathfrak{R}F} F(\gamma_0)(A)
\end{array}
\]

These maps induce (1) a $\mathfrak{R}F N$-polarization $\lambda'$ which extends the $\mathfrak{R}F F(a)$-polarization structure of $\lambda$ (2) an $F$-algebra map $i' : \mathfrak{R}F N \hookrightarrow \text{End}_{\mathfrak{R}F} F(a)$ extending $\mathfrak{R}F F(a) \hookrightarrow \text{End}_{\mathfrak{R}F} F(\gamma_0)(A)$ given by $i$ and $a$, and (3) a Hermitian $\mathfrak{R}F N$-module structure on $V$ with the pairing $\langle \cdot, \cdot \rangle$ which extends the Hermitian $\mathfrak{R}F F(\gamma_0)$-module structure given by $\gamma_0$. So the invariant $\alpha(N, V; (A, \lambda', i')) \in X^*(\hat{T}^\Gamma)$ makes sense.

We claim that $\alpha(N, V; (A, \lambda', i'))$ maps to $\alpha(\gamma_0; (A, \lambda, i), [a]) \in X^*(Z(\hat{I}_0)^\Gamma)$ via the inclusion $Z(\hat{I}_0) \hookrightarrow \hat{T}$. Keeping the compatibility (36) in mind, let us verify that $\alpha_v(N, V; (A, \lambda', i'))$ is sent to $\alpha_v(\gamma_0; (A, \lambda, i), [a])$ for every $v$. This is clear from the definition when $v = \infty$. For $v \neq p, \infty$, this follows from the functoriality of the map $\alpha_{(\cdot), v}$ in Lemma 2.3. For $v = p$, we use the functoriality of the map $\kappa_{(\cdot)}$ in §4. In other words, we appeal to the following commutative diagrams for $v \neq \infty, p$ and $v = p$, respectively.

\[
\begin{array}{ccc}
H^1(\mathbb{Q}_v, T) & \longrightarrow & H^1(\mathbb{Q}_v, I_0) \\
\alpha_{T,v} & & \alpha_{I_0,v} \\
X^*(\hat{T}^\Gamma(v)) & \longrightarrow & X^*(Z(\hat{I}_0)^\Gamma(v)) \\
\kappa_T & & \kappa_{I_0} \\
X^*(\hat{T}^\Gamma(p)) & \longrightarrow & X^*(Z(\hat{I}_0)^\Gamma(p))
\end{array}
\]

As a result of the above claim, it suffices to prove that $\alpha(N, V; (A, \lambda', i'))$ is trivial. By Lemma 11.1, we can find a $\mathfrak{R}F N$-polarization $\lambda_0$ and a Hermitian $\mathfrak{R}F N$-module $W = W_0$ satisfying the conditions in that lemma. Thus the invariants $\alpha(N, W; (A, \lambda', i'))$ and $\alpha(N, W; (A, \lambda_0, i'))$ may be considered. As further reduction steps, we make two claims, which are very similar to the ones found in the proof of [Kot92, Lem 13.2].
The first claim is that \( \alpha(N,V;(A,\lambda,i')) = \alpha(N,W;(A,\lambda',i')) \). To prove this, denote by \( t \) the element of \( H^1(\mathbb{Q},T) \) measuring the difference of \( V \) and \( W \) as Hermitian \( B \otimes_F N \)-modules. For each place \( v \), write \( t_v \) for the image of \( t \) under \( H^1(\mathbb{Q},T) \to H^1(\mathbb{Q}_v,T) \to X^*(\hat{T}^F(v)) \). This amounts to choosing an \( M \)-algebra embedding \( F^{-}\text{adic type over } v \) for the image of \( \gamma_0 \). The second claim is that \( \alpha_v(N,W;(A,\lambda,0,i')) = \alpha_v(N,W;(A,\lambda_0,i')) \). Observe that \( T \) is isomorphic to \( H_{\lambda'} \) defined by \( H_{\lambda'}(\mathbb{Q}) = \{ g \in \text{End}_{B \otimes_F N}(A) | g v^{i'} \in \mathbb{Q}^* \} \). We write \( t \) for the element of \( \ker(H^1(\mathbb{Q},H_{\lambda'}) \to H^1(\mathbb{R},H_{\lambda'})) \) measuring the difference between \( \lambda' \) and \( \lambda_0 \) (Lemma 9.2). Let \( t_v \) be the image of \( t \) under \( H^1(\mathbb{Q},T) \to H^1(\mathbb{Q}_v,T) \to X^*(\hat{T}^F(v)) \). Then \( \alpha_v(N,W;(A,\lambda_0,i')) = \alpha_v(N,W:(A,\lambda,i')) ) \) for every place \( v \) (including the case \( v = \infty \) where both sides are trivial). Therefore an application of Lemma 2.3 as before shows that \( \alpha(N,W;(A,\lambda,i')) = \alpha(N,W;(A,\lambda_0,i')) \).

As a result of the previous claims, we only need to prove that \( \alpha(N,W;(A,\lambda_0,i')) \) is trivial. This is exactly Corollary 11.3.

\[ \square \]

**Corollary 12.2.** Suppose that \( (\gamma_0; \gamma, \delta) \in KT_b \) corresponds to \( (A,i) \) via Lemma 10.4 and Corollary 8.5. Let \( \hat{i} : B \otimes_F F(\gamma_0) \hookrightarrow \text{End}_B^l(A) \) be an extension of \( i \) and \( \lambda \) be a \( B \otimes_F F(\gamma_0) \)-polarization of \( A \) with respect to \( \hat{i} \). Then \( \alpha(\gamma_0; (A,\lambda,\hat{i})) \) and \( \beta(\gamma_0; (A,\lambda,i)) \) are trivial.\n
**Proof.** As in the second through the fourth paragraph of the proof of the last lemma, we can find \( \alpha(N,V;(A,\lambda,i')) \) which maps to \( \alpha(\gamma_0; (A,\lambda,\hat{i})) \). To do so, it is enough to replace \( a \) by \( \gamma_0 \) in the argument. (So \( J = \mathcal{Z}_{\mathcal{H}}(\gamma_0) \), \( N \) is the centralizer of \( T(\mathbb{Q}) \) in \( \text{End}_B^l(\mathbb{Q})(A) \), etc.) The triviality of \( \alpha(N,V;(A,\lambda,i')) \) is proved in the same way as in the last lemma. Therefore \( \alpha(\gamma_0; (A,\lambda,\hat{i})) \) is trivial.\n
Similarly we repeat the argument of the last lemma to find \( \alpha(N,V;(A,\lambda,i')) \) which maps to \( \beta(\gamma_0; (A,\lambda,i)) \) under the natural map \( X^*(\hat{T}^F) \to X^*(\mathcal{H}_0)^{\mathbb{R}} \). To see this, replace \( J_0, I, F_0, F(\gamma_0) \) by \( H_0, \mathcal{H}_{(A,\lambda,i)} \), \( M, \mathcal{M} \) in the proof of Lemma 12.1, respectively. Again, \( \alpha(N,V;(A,\lambda,i')) \) is proved to be trivial in the same way. The proof is complete.\n
\[ \square \]

The next lemma will play a crucial role in rewriting the counting point formula in a group-theoretic way.

**Lemma 12.3.** The map in Lemma 12.1 defines a set bijection from \( FP_{A,V}^b \) onto \( KT_b^{\text{eff}} \).

**Proof.** We will construct the backward map from \( KT_b^{\text{eff}} \) to \( FP_{A,V}^b \) which is inverse to the map in Lemma 12.1. So our starting point is a Kottwitz triple \( (\gamma_0; \gamma, \delta) \) with \( \alpha(\gamma_0; \gamma, \delta) \) being trivial.

Suppose that \( (\gamma_0; \gamma, \delta) \) is mapped to a minimal \( p \)-adic type \( (M, \tilde{g}, \tilde{i}) \) under the map in Lemma 10.4. This \( p \)-adic type over \( F \) naturally corresponds to a pair \( (A,i) \) as in (i) of Corollary 8.5. We choose an \( M \)-algebra embedding \( F(\gamma_0) \hookrightarrow \text{End}_B^l(A) \) whose existence is guaranteed since the embedding exists locally at every place (use (23)). This amounts to choosing an \( F \)-algebra embedding \( i' : B \otimes_f F(\gamma_0) \hookrightarrow \text{End}_B^l(A) \). With respect to \( i' \), there exists a \( B \otimes_F F(\gamma_0) \)-polarization \( \lambda_1 \) by Lemma 9.2.

Our plan is to find a polarization \( \lambda \) such that \( ((A,\lambda,i)) \in \text{PIC}_b \) and an element \( a \in H_{(A,\lambda,i)}(\mathbb{Q}) \) such that \( ((A,\lambda,i), [a]) \) belongs to \( FP_{A,V}^b \) and maps to \( (\gamma_0; \gamma, \delta) \) via the map of Lemma 12.1. First off, we will search for a \( B \otimes_F M \)-polarization \( \lambda \). For this we will use the fact that \( \beta(\gamma_0; \gamma, \delta) \) is trivial. The last fact is an immediate consequence of the fact that \( \alpha(\gamma_0; \gamma, \delta) \) is trivial.

Let \( H_1 \) be the \( \mathbb{Q} \)-algebraic group with \( H_1(\mathbb{Q}) = \{ g \in \text{End}_B^l(A) | g^{i_1} \in \mathbb{Q}^* \} \).
Note that $H_1$ is an inner form of $H_0$ as we may argue as in [Kot92, p.424]. (Recall the definition of $H_0$ from (27).) For each $v$, let $d_v = \beta_v(\gamma_0; \gamma, \delta) - \beta_v(\gamma_0; (A, \lambda, i))$ as an element of $X^*(\mathbb{Z}(\hat{H}_0)^{F(v)}) = X^*(\mathbb{Z}(\hat{H}_1)^{F(v)})$. In fact, each $d_v$ for $v \neq \infty$ is trivial on the connected component of $\mathbb{Z}(\hat{H}_0)^{F(v)}$ and may be viewed as an element of $A_v(H_0)$. Since $d_\infty$ is trivial by the definition of $\beta_\infty(\gamma_0; (A, \lambda, i))$, we will view $d_\infty$ as the trivial element of $A_\infty(H_0)$. We know that both $\beta(\gamma_0; \gamma, \delta)$ and $\beta(\gamma_0; (A, \lambda, i))$ are trivial in $X^*(\mathbb{Z}(\hat{H}_0)^{F})$. Indeed, the former is trivial by assumption and the latter by Corollary 12.2. So their difference is trivial, namely the image of $(d_v)_v$ under $H^1(\mathbb{Q}, H_1(\mathfrak{F})) \to \oplus_v A_v(H_1) \to A(H_1)$ is trivial. We deduce from Lemma 2.3 that there exists $d \in H^1(\mathbb{Q}, H_1(\mathfrak{F}))$ mapping to $(d_v)_v$ in $H^1(\mathbb{Q}, H_1(\mathfrak{F}))$. In particular, we have $d \in \ker(H^1(\mathbb{Q}, H_1) \to H^1(\mathbb{R}, H_1))$, which implies by the argument in the proof of [HT01, Lem V.4.3] that there exists a $B \otimes_F M$-polarization $\lambda$ on $A$ (with respect to $d|_{B \otimes_F M}$) such that for all $v \neq \infty$, $\beta_v(\gamma_0; \gamma, \delta) - \beta_v(\gamma_0; (A, \lambda, i))$ is trivial. The last condition can be interpreted as the existence of the following isomorphisms.

\[
\begin{align*}
V \otimes \mathcal{A}^\infty_p & \simeq (V_p A)_\lambda \\
\forall (\Sigma, \lambda, i, \Sigma) & \simeq \forall (A[p^\infty], \lambda, i) \quad \text{as isocrystals with } B \otimes_F M \otimes \mathbb{Q}_p \text{-action}
\end{align*}
\]

which preserve Hermitian pairings up to $(\mathcal{A}^\infty_p)^\times$ and $L^\times$, respectively. This proves that $(A, \lambda, i)$ represents an element of $	ext{PIC}_B$.

Next we construct an element $a \in H(A, \lambda, i)(\mathbb{Q})$. Recall that we have a $B \otimes_F F(\gamma_0)$-polarization $\lambda_1$ with respect to $i'$. Let $I_1$ be the $\mathbb{Q}$-algebraic group with

\[I_1(\mathbb{Q}) = \{ g \in \text{End}^0_B(\mathbb{Q})(A)| g^{1, \lambda_1} g \in \mathbb{Q}^\times \}.
\]

As it was the case for $H_1$ and $H_0$, we see that $I_1$ is an inner form of $I_0$. We basically repeat the argument in the last paragraph, replacing $B \otimes_F M$ with $B \otimes_F F(\gamma_0)$ and $H_1$ with $I_1$. When $v \neq \infty$, let $e_v := \alpha_v(\gamma_0; \gamma, \delta) - \alpha_v(\gamma_0; (A, \lambda, i'))$, which may be viewed as elements of $A_v(I_1)$ for $A_v(I_1)$. Let $e_\infty \in A_\infty(I_1)$ be the trivial element. We know that both $\alpha_v(\gamma_0; \gamma, \delta)$ and $\alpha_v(\gamma_0; (A, \lambda, i'))$ are trivial by the initial assumption and Corollary 12.2, respectively. As in the last paragraph, we can find $e \in \ker(H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{R}, I_1))$ which maps to $(e_v)_v$, under $H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}, I_1(\mathfrak{F}))$. Moreover, $e$ can be chosen so that $e$ maps to $d$ under $H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}, H_1)$. This can be seen from the following commutative diagram coming from Lemma 2.3. The left vertical map is surjective by [HT01, p.174].

\[
\begin{array}{cccccc}
1 & \to & \ker^1(\mathbb{Q}, I_1) & \to & H^1(\mathbb{Q}, I_1) & \to & H^1(\mathbb{Q}, I_1(\mathfrak{F})) & \to & A(I_1) \\
& \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \ker^1(\mathbb{Q}, H_1) & \to & H^1(\mathbb{Q}, H_1) & \to & H^1(\mathbb{Q}, H_1(\mathfrak{F})) & \to & A(H_1)
\end{array}
\]

The cocycle $e$ naturally corresponds to a $B \otimes_F F(\gamma_0)$-polarization $\lambda'$ (with respect to $i'$) by Lemma 9.3. The following properties of $\lambda'$ result from the construction of $e$.

(i) $V \otimes \mathcal{A}^\infty_p$ and $(V^p A)_\lambda'$ are equivalent as $B \otimes_F F(\gamma_0) \otimes \mathbb{Q} \otimes \mathcal{A}^\infty_p$-Hermitian modules

(ii) $\forall (\Sigma, \lambda, i, \Sigma) \simeq \forall (A[p^\infty], \lambda', i')$ as isocrystals with $B \otimes_F F(\gamma_0) \otimes \mathbb{Q}_p$-action, preserving Hermitian pairings up to $L^\times$

(iii) $\lambda'$ is equivalent to $\lambda$ as $B \otimes_F M$-polarizations (via $B \otimes_F M \hookrightarrow B \otimes F F(\gamma_0)$)

The part (iii) implies that there exists $h \in \text{End}^0_B(\mathbb{Q})(A)^\times$ such that $h^\lambda' h^{-1} = \gamma \lambda$ for some $\gamma \in \mathbb{Q}^\times$. Then the association $g \mapsto hg^{-1}$ defines an $M$-algebra map $\text{End}^0_B(\mathbb{Q}) \rightarrow \text{End}^0_B(\mathbb{Q})$ compatible with involutions $\frac{1}{2} \lambda$ and $\frac{1}{2} \lambda'$. The fact that $\lambda'$ is a $B \otimes_F F(\gamma_0)$-polarization means that the map $i'$ induces
an $M$-algebra map $F(\gamma_0) \hookrightarrow \text{End}_{12}(A)$ compatible with involutions $c$ and $\overline{\chi}_x$. Finally we define $a := h^{-1}i(\gamma_0)h$. Then $a^{M}a = \gamma_0[\gamma_0] \in Q^\times$. Hence $a \in H_{(A,\lambda,i)}(Q)$, which is the desired element.

Now that we have explained how to associate $((A,\lambda,i),[a])$ to a Kottwitz triple $(\gamma_0; \gamma, \delta)$ whose Kottwitz invariant vanishes, we need to verify that $((A,\lambda,i),[a])$ is an element of $FP_{b}^{AV}$. For this, (37) and (38) tell us that $(A,\lambda,i)$ represents an element of $\text{PIC}_{b}$, and the acceptability of $a \in H_{(A,\lambda,i)}(Q)$ is inherited from $\delta$. The well-definedness of $((A,\lambda,i),[a])$ is easy to check. By changing $\gamma$ and $\delta$ in their conjugacy classes we change $(A,\lambda,i)$ within its equivalent class and $a$ within its $\kappa$-conjugacy class. Although replacing conjugacy classes we change $(\gamma_0; \gamma, \delta)$, the well-definedness of $((A,\lambda,i),[a])$ is unchanged.

Finally, we verify that the map constructed above is the inverse of the map in Lemma 12.1. Surjectivity follows from our construction. If $(\gamma_0; \gamma, \delta)$ maps to $((A,\lambda,i),[a])$ then it is readily checked that $i((A,\lambda,i),[a])$ is conjugate to $(\gamma, \delta)$ in $G(\mathbb{A}^{\infty,p}) \times J_b(Q_p)$. Since the stable conjugacy class of $\gamma_0$ is determined by the conjugacy class of $(\gamma, \delta)$, the image of $a$ under the map in Lemma 12.1 is stably conjugate to $\gamma_0$ in $G(Q)$.

To see injectivity, suppose that both $((A,\lambda,i),[a])$ and $((A',\lambda',i'),[a'])$ map to $(\gamma_0; \gamma, \delta)$. Let $z := [(A,\lambda,i)]$ and $z' = [(A',\lambda',i')]$. Consider the minimal $p$-adic type over $F$ for $(A,i)$, defined on the $F$-algebra $M_z$ (see Proposition 8.4). Using the $F$-algebra embedding $\zeta : M_z \hookrightarrow F(a)$ in Lemma 8.6, get an equivalent $p$-adic type $(F(a), (\eta_i), (\eta'_i))$. We claim that the last $p$-adic type is equivalent to the $p$-adic type constructed from $(\gamma_0; \gamma, \delta)$ in Lemma 10.4 under an isomorphism $F(a) \simeq F(\gamma_0)$ taking $a$ to $\gamma_0$. This is so because $(\eta_i)$ and $(\eta'_i)$ are determined by the valuation of $a_i$ at each place $p$ of the fields $F(a_i)$, if we write $F(a) = \prod F(a_i)$ as a product of fields and denote by $(a_i)$ the image of $a$. The point is that since $a$ is acceptable, the $p$-adic valuation of $a_i$ recovers the slope of the part of $A[p^\infty]$ on which $a_i$ acts. So $(\eta_i)$ and $(\eta'_i)$ can be recovered in view of (ii) of Corollary 8.5. In fact, we constructed a $p$-adic type from $(\gamma_0; \gamma, \delta)$ using the $p$-adic valuations of $\gamma_0$ in the fields $F_i$ where $F(\gamma_0) = \prod F_i$ is a decomposition into fields. This proves our claim. As a consequence, the $p$-adic type for $(A,i)$ is equivalent to the one for $(A',i')$ since both are equivalent to the one constructed from $(\gamma_0; \gamma, \delta)$. It is easy to verify that $(A,\lambda,i)$ and $(A',\lambda',i')$ are nearly equivalent using the earlier part of the current proof involving $\beta_p$-invariant. Therefore there is an isomorphism $H_2 \simeq H_2'$ by Lemma 9.6, which is well-defined up to $Q$-conjugacy. It remains to see that $a$ and $a'$ are $H_2(\mathbb{A})$-conjugate via this isomorphism. Without loss of generality we may assume that $(A',\lambda',i') = (A,\lambda,i)$. Since $C$-conjugacy and $R$-conjugacy coincide in $H_2(R)$, it suffices to check that $a$ and $a'$ are $H_2(\mathbb{A}^{\infty})$-conjugate. By Lemma 3.3, $a$ and $a'$ are $H_2(\mathbb{A}^{\infty})$-conjugate if and only if the Hermitian $B \otimes_F F(a) \otimes Q \mathbb{A}^{\infty,p}$-module $(V^p A)_\lambda$ and the Hermitian $B \otimes_F F(a') \otimes Q \mathbb{A}^{\infty,p}$-module $(V^p A)_{\lambda}$ are equivalent via $F(a) \simeq F(a')$ with $a \mapsto a'$. Similarly, $a$ and $a'$ are $H_2(Q_p)$-conjugate if and only if the isocrystal $V(A[p^\infty], \lambda,i)$ with the Hermitian $B \otimes_F F_{Q_p}(a)$-pairing is isomorphic to the isocrystal $V(A[p^\infty], \lambda,i)$ with the Hermitian $B \otimes_F F_{Q_p}(a')$-pairing such that the two pairings match up to $L^\times$. (The last fact, an analogue of (part of) Lemma 3.3, can be proved analogously as that lemma.) But we know that $i_\sigma(a)$ and $i_\sigma(a')$ are conjugate in $G(\mathbb{A}^{\infty,p}) \times J_b(Q_p)$ since they are conjugate to $(\gamma, \delta)$ therein. This implies that the two Hermitian modules above are equivalent and the two isocrystals above are isomorphic with additional structure.(Apply Lemma 3.3 and its analogue for isocrystals again.) Therefore $a$ and $a'$ are $H_2(\mathbb{A}^{\infty})$-conjugate.

\[\square\]

13 Final form of the counting point formula

We go back to the analysis of cohomology of Igusa varieties. Assuming that $\varphi \in C^{\infty}_c(G(\mathbb{A}^{\infty,p}) \times J_b(Q_p))$ is an acceptable function, we combine Lemma 7.4 and Lemma 8.6 to obtain the following
expression for \( \text{tr}(\varphi|_{H_c(Ig_0, \mathcal{L}_z)}) \).

\[
\sum_{z \in P_{rH_b} \mathcal{H}} \sum_{\mathcal{H}(a) \in H_c(Q) / \sim} \text{vol}(\iota_z(Z_{H_c}(a)(Q)) \langle \iota_z(Z_{H_c}(a)(H_c)) \rangle \cdot \text{tr} \xi(\iota_z(a)) \cdot O_{I_{\varepsilon}^{\pi}}(\varphi).
\]

In view of Lemma 9.6, the summand in the above sum depends only on \( H_c(\mathcal{H}) \)-conjugacy class of \( a \) and near equivalence class of \( (A, \lambda, i) \). Thus we merge terms to rewrite the sum over the set \( FP^{AV}_{\varepsilon} \). Proceeding exactly as in the proof of [HT01, Lem V.3.3] (but keeping the expression \(|A(Z_{H_c}(a))|\) and not changing it into \( \kappa_B \) in their notation), we arrive at the expression (39). In the equality \( Z_{H_c}(a)(R)^1 \) denotes the kernel of the map \( Z_{H_c}(a)(R) \rightarrow R_{>0}^\infty \) given by \( x \mapsto |x^{1/2}x|_R \). For the choice of appropriate Haar measures on \( Z_{H_c}(a) \), one may read Theorem 13.1 below, replacing \( I_0 \) with \( Z_{H_c}(a) \).

\[
\text{tr}(\varphi|_{H_c(Ig_0, \mathcal{L}_z)}) = \sum_{(z, [a]) \in FP^{AV}_{\varepsilon}} \text{vol}(Z_{H_c}(a)(R)^1 \langle A(Z_{H_c}(a)) \rangle \cdot \text{tr} \xi(\iota_z(a)) \cdot O_{I_{\varepsilon}^{\pi}}(\varphi) (39)
\]

Recall that \( I_0 = Z_G(\gamma_0) \) as usual. The \( \mathbb{R} \)-group \( I_\infty \) denotes the inner form of \( I_0 \) over \( \mathbb{R} \) which is compact modulo center. (In fact, \( I_\infty(\mathbb{R}) \simeq Z_{H_c}(a)(R) \) since both are compact modulo center inner forms of \( I_0 \) over \( \mathbb{R} \).) We define \( I_0(\mathbb{A})^1 \) to be the kernel of \( I_0(\mathbb{A}) \rightarrow \mathbb{R}_{>0}^\infty \) given by \( x \mapsto |x^{1/2}x|_{\mathbb{A}} \). Applying Lemma 12.3, we rewrite (39) in terms of Kottwitz triples to obtain the final result.

**Theorem 13.1.** If \( \varphi \in C_c^\infty(G(\mathcal{A}_\infty^\delta) \times J_0(\mathbb{Q}_p)) \) is acceptable, then

\[
\text{tr}(\varphi|_{H_c(Ig_0, \mathcal{L}_z)}) = \sum_{(\gamma_0, \gamma, \delta) \in KT^\infty_{\varepsilon}} \text{vol}(I_\infty(\mathbb{R})^1 \langle A(I_0) \rangle \cdot \text{tr} \xi(\gamma_0) \cdot O_{I_{\varepsilon}^{\pi}}(\varphi)
\]

with the following choice of Haar measures. Choose the Tamagawa measure on \( I_0(\mathbb{A})^1 \). Choose Haar measures on \( I_0(\mathbb{A})^1 \) and \( I_0(\mathbb{R})^1 \) compatibly with the measure on \( I_0(\mathbb{A})^1 \) via the exact sequence

\[
1 \rightarrow I_0(\mathbb{R})^1 \rightarrow I_0(\mathbb{A})^1 \rightarrow I_0(\mathbb{A}_\infty) \rightarrow 1.
\]

We define Haar measures on \( Z_G(\gamma_i)(\mathbb{Q}_v) \) \((v \neq p, \infty)\), \( I_0(\mathbb{Q}_p) \) and \( I_\infty(\mathbb{R})^1 \) compatibly with those on \( I_0(\mathbb{Q}_v) \), \( I_0(\mathbb{Q}_p) \) and \( I_0(\mathbb{R})^1 \), respectively (i.e. compatible choice of measures on inner forms in the sense of [Kot88, p.631]).

**Remark 13.2.** The only implicit assumption for the above theorem is that the Igusa variety \( Ig_0 \) should arise from an unramified integral PEL datum (Definition 5.2). In particular \( G \) should be unramified over \( \mathbb{Q}_p \).

**Remark 13.3.** It is worth noting that our formula is very similar to Kottwitz’s formula [Kot92, (19.5)] and indeed inspired by it. However there is no naive explicit relation between the two formulas. Observe that our orbital integrals at \( p \) are quite different from those of Kottwitz and that the triples \((\gamma_0, \gamma, \delta)\) have a somewhat different meaning.

**References**


