18.786. On HW 7, problem 6

The only subtle part is the surjectivity, namely that G_0 and G_1 map *onto* G'_0 and G'_1, respectively. Let me write $G' = G/H$ so that $G'_0 = (G/H)_0$ and $G'_1 = (G/H)_1$. Put $n := |G|$ and $e_H := |H_0|$(=ramification index for L/L^H). I write K' for the intermediate field L^H. I can think of two proofs for surjectivity.

First proof. For $\overline{\sigma} \in G/H$ and any lift $\sigma \in G$, we have

$$i_{G/H}(\overline{\sigma}) = \frac{1}{e_H} \sum_{\tau \in H} i_G(\sigma\tau).$$

If $\overline{\sigma} \in (G/H)_0$ then $i_{G/H}(\overline{\sigma}) \geq 1$. Hence $\exists \tau \in H$, $i_G(\sigma\tau) \geq 1$. For such a τ, the element $\sigma\tau$ belongs to G_0 and maps to $\overline{\sigma}$.

Now suppose $\overline{\sigma} \in (G/H)_1$ so that $i_{G/H}(\overline{\sigma}) \geq 2$. By the $(G/H)_0$ case, we may arrange that $i_G(\sigma) \geq 1$. If $\tau \notin H_0$, then $i_H(\tau) = 0$ so $i_G(\tau) = 0$. Since $i_G(\sigma) > i_G(\tau)$, we have $i_G(\sigma\tau) = \min(i_G(\sigma), i_G(\tau)) = 0$. Hence

$$2 \leq \frac{1}{e_H} \sum_{\tau \in H} i_G(\sigma\tau) = \frac{1}{|H_0|} \sum_{\tau \in H_0} i_G(\sigma\tau).$$

It follows that $\exists \tau \in H_0$ s.t. $i_G(\sigma\tau) \geq 2$. Such a $\sigma\tau$ is the desired element. $\square$

Sketch of second proof. Basically one can show that L^{G_0} (resp. L^{G_1}) are the unique maximal unramified (resp. at most tamely ramified, equivalently $e_{L/K}$ is prime to the residue field characteristic[1]) extension of K in L. (In fact this remains true for infinite Galois extensions.) From this it is easy to deduce that G_0 and G_1 surject onto G'_0 and G'_1, resp: the assertion amounts to saying that the maximal unramified (resp. at most tamely ramified) extension of K in L contains the analogous extension of K in K', which is obviously true since $K' \subset L$. $\square$

[1]as usual assume that the residue field extension k_L/k_K is separable