2 Let K be a number field with an algebraic closure $\overline{K}$. Let L and L' be finite extensions of K in $\overline{K}$. Prove or disprove: If a prime $\mathfrak{p}$ of $\mathcal{O}_K$ is totally ramified in L and L' then it is totally ramified in the composite field LL'.

Example 1. Let p be an odd prime and take $L = \mathbb{Q}(\sqrt{p})$, $L' = \mathbb{Q}(\sqrt{-p})$. Clearly p is totally ramified in L and L'. Since p is unramified in the subextension $\mathbb{Q}(i)$ of LL', it is not totally ramified in LL'.

Example 2. Let $p = 2$, $L = \mathbb{Q}(\sqrt[3]{2})$, $L' = \mathbb{Q}(\sqrt[3]{2} \cdot \zeta_3)$. Then p is totally ramified in L and L' but not in the subextension $\mathbb{Q}(\zeta_3)$ of LL'.

4 Let $n \geq 3$. We have seen that a prime p is ramified in $\mathbb{Q}(\zeta_n)$ if and only if $p|n$. (You need not prove this.) Describe the set of all primes p which split completely in $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ in terms of a congruence condition on p.

When K is a finite Galois extension of $\mathbb{Q}$ in which p is unramified, let $\mathrm{Frob}_{K,p} \in \mathrm{Gal}(K/\mathbb{Q})$ denote the Frob element. Then one proves

- $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is a (totally real) subfield of $\mathbb{Q}(\zeta_n)$ of index 2 fixed under $\{1, c\} \subset \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, where c is the complex conjugation (w.r.t any embedding $\mathbb{Q}(\zeta_n) \hookrightarrow \mathbb{C}$).
- $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1})) \simeq \{1, c\}$ maps to $\{\pm 1\}$ under the canonical isom

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times, \qquad \forall p \nmid n,\ \mathrm{Frob}_{\mathbb{Q}(\zeta_n),p} \leftrightarrow p.$$

- The natural projection $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n + \zeta_n^{-1})/\mathbb{Q})$ carries $\mathrm{Frob}_{\mathbb{Q}(\zeta_n),p}$ to $\mathrm{Frob}_{\mathbb{Q}(\zeta_n+\zeta_n^{-1}),p}$.

This implies that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n + \zeta_n^{-1})/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}, \qquad \forall p \nmid n,\ \mathrm{Frob}_{\mathbb{Q}(\zeta_n+\zeta_n^{-1}),p} \leftrightarrow p.$$

One concludes that p splits completely in $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$

$\Leftrightarrow$ $\mathrm{Frob}_{\mathbb{Q}(\zeta_n+\zeta_n^{-1}),p}$ is trivial $\Leftrightarrow$ $p \equiv \pm 1 \pmod{n}$.