18.786. (Fall 2011) Midterm Exam, Tuesday Oct 25, 11-12:30

* Each problem is worth 20 points. Write your solutions clearly.

1. Give the definitions of the following. (Any one of the equivalent definitions is fine.) Five points for each item.
   (a) a Dedekind domain
   (b) a fractional ideal of K, which is the fraction field of a Dedekind domain A
   (c) a complete lattice in a finite dimensional $\mathbb{R}$-vector space (with the usual topology)
   (d) the inertia group of $\mathfrak{P}$ over K, where L is a finite Galois extension of a number field K and $\mathfrak{P}$ is a nonzero prime ideal of $\mathcal{O}_L$

2. Let K be a number field with an algebraic closure $\overline{K}$. Let L and L' be finite extensions of K in $\overline{K}$. Prove or disprove: If a prime $\mathfrak{p}$ of $\mathcal{O}_K$ is totally ramified in L and L' then it is totally ramified in the composite field LL'.

3. Let K be a number field with ring of integers $\mathcal{O}_K$, and L be a finite Galois extension over K. Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_K$. Prove that the Galois group $\mathrm{Gal}(L/K)$ acts transitively on the set of prime ideals of $\mathcal{O}_L$ containing $\mathfrak{p}$.

4. Let $n \geq 3$. We have seen that a prime p is ramified in $\mathbb{Q}(\zeta_n)$ if and only if $p|n$. (You need not prove this.) Describe the set of all primes p which split completely in $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ in terms of a congruence condition on p.

5. Find the class number of $\mathbb{Q}(\sqrt{7})$ and the group of units of its ring of integers (with a fundamental unit). Proof should be provided. Feel free to use the following fact: In every ideal class of an algebraic number field K of degree n, there exists an integral ideal $\mathfrak{a}$ such that

$$\mathfrak{N}(\mathfrak{a}) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}.$$

   (Here s is the number of complex places.)