18.786. On lattices

Here is a leftover from the Thursday class.

Proposition 0.1. *Let* $\Gamma' \subset \Gamma$ *be complete lattices in* V *with* $[\Gamma : \Gamma'] < \infty$. *Then*

$$\frac{\mathrm{vol}(\Gamma')}{\mathrm{vol}(\Gamma)} = (\Gamma : \Gamma').$$

Sketch of proof. Choose a large enough $R > 0$ and consider a ball $B_R = B(0; R)$ of radius R and centered at 0. Let

$$m := (\Gamma : \Gamma')$$

and choose a set of representatives $\{\gamma_1, ..., \gamma_m\}$ for Γ/Γ', which you can find in the fundamental mesh $\mathcal{F}(\Gamma')$ (any point in V can be translated by an element of Γ' into $\mathcal{F}(\Gamma')$). Since γ_i are close to 0, if $R \gg 0$ then

$$\frac{|\Gamma \cap B_R|}{|\Gamma' \cap B_R|} = \frac{\sum_{i=1}^{m} |((\gamma_i + \Gamma') \cap B_R)|}{|\Gamma' \cap B_R|} \sim \frac{\sum_{i=1}^{m} |(\Gamma' \cap B_R)|}{|\Gamma' \cap B_R|} = m.$$

On the other hand,

$$\frac{\mathrm{vol}(\Gamma')}{\mathrm{vol}(\Gamma)} = \frac{\mathrm{vol}(B_R)/\mathrm{vol}(\Gamma)}{\mathrm{vol}(B_R)/\mathrm{vol}(\Gamma')} \sim \frac{|\{\gamma \in \Gamma : \gamma + \mathcal{F}(\Gamma) \subset B_R\}|}{|\{\gamma' \in \Gamma' : \gamma' + \mathcal{F}(\Gamma') \subset B_R\}|} \sim \frac{|\Gamma \cap B_R|}{|\Gamma' \cap B_R|}.$$

In the first approximation above, use the fact that such $\gamma + \mathcal{F}(\Gamma)$ cover B_R modulo a thin layer (of constant thickness) around the boundary of B_R without overlap. The error term for each $\sim$ is bounded by $O(R^{n-1})$ (which is ignorable relative to the n-dimensional ball $\mathrm{vol}(B_R)$), and one concludes the proof by taking $R \to \infty$.

□