18.786. Uniformizer as a generator in a totally ramified extension

Let L be a finite Galois extension of a CDVF K with separable residue field extension. Let $\mathcal{O}_L$ and $\mathcal{O}_K$ denote the rings of integers. Normalize $v_L : L^\times \to \mathbb{Z}$ so that uniformizers map to 1. Put $n = [L : K]$.

Proposition 0.1. *If L is totally ramified over K then for any uniformizer π_L of L,*

$$\mathcal{O}_L = \mathcal{O}_K[\pi_L].$$

Remark 0.2. This is proved in Serre, I, §6, Prop 18 but the proof appears as part of a broader discussion. In the present context a simpler argument (based on the same idea) can be given.

Proof. Consider

$$b = a_0 + a_1\pi_L + \cdots a_{n-1}\pi_L^{n-1}, \quad a_i \in K.$$

Since $v_L(a) = e_{L/K}v_K(a) \in e_{L/K}\mathbb{Z} = n\mathbb{Z}$ for all nonzero $a \in K$, we have $v_L(a_i\pi_L^i) \in i + n\mathbb{Z}$ for $i = 0, ..., n-1$ such that $a_i \neq 0$. As these are distinct, it is clear that $a \neq 0$ unless $a_0 = a_1 = \cdots = a_{n-1} = 0$. Hence $\{1, \pi_L, ..., \pi_L^{n-1}\}$ is a basis for L over K.

By the latter fact, any $b' \in L$ can be written as

$$b = a'_0 + a'_1\pi_L + \cdots + a'_{n-1}\pi_L^{n-1}.$$

Since $a'_i\pi_L^i$ have distinct valuations (if nonzero), the non-archimedean property tells us that

$$v_L(b) = \min_{0 \le i \le n-1} v_L(a'_i\pi_L^i) = \min_{0 \le i \le n-1} (v_L(a'_i) + i).$$

So $b \in \mathcal{O}_L \Leftrightarrow \min_{0 \le i \le n-1}(v_L(a'_i) + i) \geq 0 \Leftrightarrow \min_i v_L(a'_i) \geq 0 \Leftrightarrow a'_i \in \mathcal{O}_K$ for all $0 \leq i \leq n-1$. (For the second equivalence, use $v_L(a'_i) \in n\mathbb{Z}$.) This shows that

$$\mathcal{O}_L \subset \mathcal{O}_K[\pi_L].$$

As the other inclusion is clear, we are done. □