18.786. Some clarifications

For a lattice X of a K-vector space V, we have shown (cf. Serre Prop III.1.2)

Proposition 0.1. $\chi(X, uX) = (\det u)$ *for* $u \in \mathrm{Aut}_K(V)$.

The idea was to reduce to the case where A is a DVR by localization and where $u(X) \subset X$ by scaling u, as the effect of scaling is easy to keep track of in the identity. With a choice of $X \simeq A^n$ as A-modules, u is represented by an $n \times n$ matrix with entries in A. Then I explained that u can be "diagonalized" by elementary row-column operations. I should have clarified that this is *not* the usual diagonalization by a suitable choice of basis. Rather, each row-column operation multiplies u by a simple invertible matrix (which is justified since both sides of the above equality remain the same). In other words, u is made diagonal by multiplying (rather than conjugating by) a product of invertible matrices.

In the proof of

Proposition 0.2. $\mathfrak{D}_{B/A} = (f'_\beta(\beta))$, *where* $f_\beta(x) \in A[x]$ *is the minimal polynomial of* $\beta \in B$ *such that* $B = A[\beta]$,

I used the fact that $\mathrm{tr}_{L/K}(\beta^i / f'_\beta(\beta))$ equals 0 if $0 \le i \le n-2$ and 1 if $i = n-1$, where $n = [L : K]$. (See Serre's Lemma III.6.2 for this.) The heart of the proof was that for an element of L written as

$$b = \sum_{i=0}^{n-1} a_i \frac{\beta^i}{f'_\beta(\beta)}, \quad a_i \in K$$

and compute

$$\mathrm{tr}_{L/K}(b\beta^j) = \sum_{i=0}^{n-1} a_i \mathrm{tr}_{L/K}(\frac{\beta^{i+j}}{f'_\beta(\beta)}) = a_{n-1-j} + (A\text{-lin. combination of } a_{n-j}, ..., a_{n-1}).$$

(I missed the "error terms" in the first attempt; these come from the coefficients of f_β as β^n, β^{n+1}, ... are written in the basis $\{1, \beta, ..., \beta^{n-1}\}$.) From this it is easy to deduce (by induction on j) that

$$\mathrm{tr}_{L/K}(b\beta^j), \ \forall 0 \le j \le n-1 \quad \Leftrightarrow \quad a_0, ..., a_{n-1} \in A$$

and thus the codifferent $B^* = (f'_\beta(\beta)^{-1})$.