Problem 4 *(Neukirch II.4.1)*

We assume the valuation is not trivial. First note K cannot be a finite field: There is no non-trivial homomorphism from $\mathbb{F}_q^\times$ to $\mathbb{R}_{>0}^\times$, since the latter is torsion-free.

The conclusion then follows from the following cardinality calculations.

Lemma 4.1: Let L/K be an infinite algebraic extension, with K infinite. Then

$$|L| \le |K|.$$

Proof. The number of polynomials with degree n with coefficients in K is $|K^\times||K|^n = |K|^{n+1}$. Since K is infinite,

$$|K[x]| = |K| + |K|^2 + \cdots = |K|.$$

Each element of L is the root of some polynomial in $K[x]$, and each polynomial has finitely many roots, so $|L| \le |K|$. □

Lemma 4.2: Suppose L/K is an extension of complete fields and $[L : K]$ is infinite. Then

$$|L| > |K|.$$

Proof. Take an infinite number of linearly independent elements $\{a_j\}_{j\in\mathbb{N}}$ of L as a vector space over K. Note there are elements of K with arbitrarily small norm: take an arbitrary element with norm not equal to 1 and take an appropriate power. Thus by scaling we may assume $|a_{j+1}| < \frac{|a_j|}{2}$ for all j. We claim that the elements

$$s(\varepsilon) := \sum_{j=1}^{\infty} \varepsilon_j a_j$$

are distinct for all $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. (The sum is defined in L because the condition ensures that the partial sums form a Cauchy sequence.)

If $s(\varepsilon) = s(\varepsilon')$ for $\varepsilon \neq \varepsilon'$, consider

$$\sum_{j=1}^{\infty} (\varepsilon_j - \varepsilon'_j) a_j = 0.$$

Each of the coefficients is 0 or ± 1, with at least one nonzero. Take the first such index k; we have

$$|a_k| = |(\varepsilon_k - \varepsilon'_k) a_k| = \left| \sum_{j=k+1}^{\infty} (\varepsilon'_j - \varepsilon_j) a_j \right| < \sum_{j=1}^{\infty} \frac{|a_k|}{2^j} = |a_k|,$$

contradiction. (More precisely, by the triangle inequality $\left|\sum_{j=k+1}^{l} (\varepsilon_j - \varepsilon'_j) a_j\right| < |a_{k+1}| + \frac{|a_k|}{2} < |a_k|$ for any l; now take the limit as $l \to \infty$.)

The number of sums $s(\varepsilon)$ is $2^{|K|} > |K|$, so $|L| \ge 2^{|K|} > |K|$. □