18.786. (Fall 2011) Homework # 9 (due Thu Dec 08)

Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Fix an algebraic closure $\overline{K}$ of K. All formal groups are assumed to be 1-dimensional and commutative.

1. The **existence theorem** of LCFT (which especially concerns surjectivity) asserts the following: The map from the set of finite abelian extensions of K (in $\overline{K}$) to the set of open subgroups of $K^\times$ of finite index given by $L \mapsto N_{L/K}(L^\times)$ is a bijection. (You may skip proving that $N_{L/K}(L^\times)$ is open and of finite index in $K^\times$. Try it on your own, or see Serre's local fields p.172 or [CF67] p.143, Th 3, for instance.) Deduce the existence theorem from the main statements of LCFT as stated in class, cf. Theorem A of [Y].

   Remark 0.1. Notice that "open subgroups of $K^\times$" are objects intrinsic to K. (For instance, there's no need to talk about elements external to K.) This shows one aspect of CFT: to classify abelian extensions of K in terms of intrinsic data for K.

2. Consider a ring A and an A-algebra B with an ideal $\mathfrak{m}$. Suppose that B is $\mathfrak{m}$-adically complete, i.e. the canonical map from B to the inverse limit of $B/\mathfrak{m}^n$ over $n \geq 1$ is an isomorphism. Let $F \in A[[X, Y]]$ be a formal group over A. Define a binary operation $+_F$ on the set $\mathfrak{m}$ by

$$\alpha +_F \beta := F(\alpha, \beta), \quad \alpha, \beta \in \mathfrak{m}.$$

   Show that this is again an element of $\mathfrak{m}$ and that $(\mathfrak{m}, +_F)$ is an abelian group. Moreover if F' is another formal group over A and $f \in \mathrm{Hom}_A(F, F')$ then verify that f induces a group homomorphism from $(\mathfrak{m}, +_F)$ to $(\mathfrak{m}, +'_F)$.

   Remark 0.2. We may interpret $\mathfrak{m}$ as the set of "B-points" of F, whose underlying space is the formal scheme attached to $(A[[T]], (T))$. The problem says that F induces a group structure on the set of B-points of F. In fact, F can be viewed as a functor from an appropriate category of algebras to the category of groups.

3. Let $k \subset \overline{\mathbb{F}}_q$ be an extension of $\mathbb{F}_q$. Let $\varphi \in \mathrm{Gal}(k/\mathbb{F}_q)$ denote the arithmetic Frobenius $x \mapsto x^q$. Let $F \in k[[X, Y]]$ be *any* formal group over k and write F^φ for $\varphi(F) \in k[[X, Y]]$ ("Frobenius twist of F"). Show that $f(X) = X^q$ defines a homomorphism from F to F^φ.

4. (Proof of the global Kronecker-Weber theorem) Assume the following:

   (a) local K-W: $\mathbb{Q}_p^{\mathrm{ab}} = \cup_{n \geq 1} \mathbb{Q}_p(\zeta_n)$.

   (b) There is no nontrivial finite extension of $\mathbb{Q}$ unramified at every prime. (This follows from a discriminant bound. See Neukirch Theorem III.2.18 for instance.)

   Prove the global K-W theorem, namely that $\mathbb{Q}^{\mathrm{ab}} = \cup_{n \geq 1} \mathbb{Q}(\zeta_n)$, or equivalently that any finite abelian extension F of $\mathbb{Q}$ is contained in $\mathbb{Q}(\zeta_n)$ for some n. (Hint: Neukirch V.1.10; try to do as much as possible on your own.)

5. Recall that $G(\overline{K}/K)$ has profinite (Krull) topology. Topologically $W(\overline{K}/K)$ is a $\mathbb{Z}$-disjoint union of $G(\overline{K}/K)_0$-cosets $G(\overline{K}/K)_0\sigma_n$ (where σ_n is any lift of Frob_q^n, $n \in \mathbb{Z}$), where each $G(\overline{K}/K)_0\sigma_n$ is given the same topology as the profinite topology on $G(\overline{K}/K)_0$ via translation by σ_n. (It is easy to see that the topology on $W(\overline{K}/K)$ is independent of choices of σ_n's.) Show that the natural inclusion $\iota : W(\overline{K}/K) \to G(\overline{K}/K)$ is continuous and has dense image. Check that ι is not a topological isomorphism onto $\iota(W(\overline{K}/K))$, where the latter is equipped with the topology induced by that of $G(\overline{K}/K)$.

6. (A rephrase of LCFT) Using the topological isomorphism Art_K, construct a natural bijection between the following two sets

   - set of continuous characters(=group homomorphisms) $W(\overline{K}/K) \to \mathbb{C}^\times$
   - set of continuous characters $K^\times \to \mathbb{C}^\times$

   and show that it is indeed a bijection. (Note that $W(\overline{K}/K)$ is used, although $W(K^{\mathrm{ab}}/K)$ could be used as well.)

   Remark 0.3. One can show that a continuous character $G(\overline{K}/K) \to \mathbb{C}^\times$ should always have finite image whereas $K^\times \to \mathbb{C}^\times$ and $W(\overline{K}/K) \to \mathbb{C}^\times$ can have infinite images. This is another indication that $W(\overline{K}/K)$ is more natural.

   Remark 0.4. The above bijection is the "local Langlands correspondence for GL_1 over K". The presence of GL_1 is clearer if we rewrite the objects as $W(\overline{K}/K) \to GL_1(\mathbb{C})$ and $GL_1(K) \to GL(V)$ (with a one-dim complex vector space V). I am not going to state the local Langlands correspondence for GL_n over K (which started as a conjecture and became a theorem by Harris-Taylor and Henniart about 10+ years ago) but merely remark that LCFT generalizes more naturally (in retrospect; there were quite a few unsuccessful attempts) in the rephrased form as above.