18.786. (Fall 2011) Homework # 7 (due Tue Nov 15)

$*$ The following apply to Problems 1-3 and 6: Let L/K be a finite separable extension of CDVFs with ramification index e and separable residue field extensions. Normalize an additive valuation $v_L : L^\times \to \mathbb{Z}$ such that $v_L(\pi_L) = 1$ for any uniformizer $\pi_L \in \mathcal{O}_L$, and similarly v_K such that $v_K(\pi_K) = 1$. Assume that $p \neq 0$ is the characteristic of the residue fields. If L/K is Galois, write $G = \mathrm{Gal}(L/K)$ and G_i for the i-th ramification group.

1. (Serre, IV.2.4) Suppose that K contains a primitive p-th root of unity. Let L be the extension of K obtained by adjoining a p-th root of a uniformizer π_L. Show that L is a cyclic totally ramified Galois extension of K. If σ is a generator of $G = \mathrm{Gal}(L/K)$, show that $\sigma \in G_i$, $\sigma \notin G_{i+1}$, where $i = e/(p-1)$, with $e = v_L(p)/p - 1$.

2. (Serre, IV.2.5) Suppose that $p \nmid n$ and $n < pv_K(p)/(p-1)$. Let $\alpha \in K$ be an element of valuation $-n$. Show that the Artin-Schreier equation $x^p - x = \alpha$ is irreducible over K, and defines an extension L/K which is Galois with Galois group $\simeq \mathbb{Z}/p\mathbb{Z}$. (Hint: Show that if x is a root of this equation, then the other roots have the form $x + z_i$ $(0 \leq i < p)$, with $z_i \in A_L$ and $z_i \equiv i \bmod \mathfrak{p}_L$.)

3. Continuing Problem 2, show that $G_n = G$ and $G_{n+1} = \{1\}$.

4. Let $L = \mathbb{Q}_p(\zeta_{p^n})$ and $K = \mathbb{Q}_p$. Let $\mathfrak{P}$ denote the maximal ideal of $\mathcal{O}_L = \mathbb{Z}_p[\zeta_{p^n}]$. Compute $v_L(\mathcal{D}_{L/K})$ $(= \mathrm{ord}_{\mathfrak{P}}(\mathcal{D}_{L/K}))$ in at least two different ways. (One way uses the derivative of the minimal polynomial of a generator of $\mathcal{O}_L$ over $\mathcal{O}_K$. The other combines higher ramification groups with HW 6, Problem 1.)

5. (Serre, III.6.3) Suppose that A is a DVR and that B is "completely decomposed" in the sense that there are $n = [L : K]$ prime ideals of B above the prime ideal $\mathfrak{p}$ of A. Show that in order for there to exist an $x \in B$ such that $B = A[x]$, it is necessary and sufficient that $n \leq |A/\mathfrak{p}|$. (Note: When A and B are CDVRs, we remarked that such an x always exists, cf. Prop III.12.)

6. (Serre, IV.2.1) Assume that L/K is finite Galois. If G' is a quotient group of G, show that G'_0 and G'_1 are the images in G' of G_0 and G_1. Deduce from this (by passage to the limit) a definition of G_0 and G_1 when G is the Galois group of an *infinite* extension.