18.786. (Fall 2011) Homework # 6 (due Tue Nov 08)

1. (Neu II.10.2) Let $K = \mathbb{Q}_p$ and $K_n = K(\zeta)$ where ζ is a primitive p^n-th root of unity. Show that the ramification groups of K_n/K are given as follows ($s \in \mathbb{Z}$):
$$G_s = G(K_n/K), \quad s = 0,$$
$$G_s = G(K_n/K_1), \quad 1 \le s \le p-1,$$
$$G_s = G(K_n/K_2), \quad p \le s \le p^2 - 1,$$
$$\cdots$$
$$G_s = 1, \quad p^{n-1} \le s.$$
(Hint: We have shown that $(1-\zeta)^{\varphi(p^n)} = (p)$.)

2. Let F be a number field with a nontrivial valuation $|\cdot|_v$. Denote by F_v its completion. Fix algebraic closures $\overline{F}$ and $\overline{F}_v$ of F and F_v, resp. Consider any field embedding $\iota : \overline{F} \hookrightarrow \overline{F}_v$ and the map $r : \mathrm{Gal}(\overline{F}_v/F_v) \to \mathrm{Gal}(\overline{F}/F)$ given by restricting $\sigma \in \mathrm{Gal}(\overline{F}_v/F_v)$ to (an automorphism of) $\overline{F}$ via ι. Check that r is an injective group homomorphism. For any ι and ι', show that the induced r and r' are conjugate in the sense that there exists $\gamma \in \mathrm{Gal}(\overline{F}/F)$ such that $r' = \gamma r \gamma^{-1}$. (Hence the $\mathrm{Gal}(\overline{F}/F)$-conjugacy class of r is independent of the choice of ι.)

   * Note: When v is an infinite place such that $F_v \simeq \mathbb{R}$ (rather than $\mathbb{C}$), the above map defines a well-defined conjugacy class of "complex conjugation" c_v in $\mathrm{Gal}(\overline{F}/F)$. For instance if $[F : \mathbb{Q}] = n$ and $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^n$ then the full Galois group over F possesses (essentially) n complex conjugations corresponding to the real infinite places.

3. Keep the notation of the previous problem. Consider the set of F-algebra homomorphisms $\mathrm{Hom}_F(\overline{F}, \overline{F}_v)$ equipped with an action by $\mathrm{Gal}(\overline{F}_v/F_v)$ via $\sigma : \tau \mapsto \sigma \circ \tau$. Let $|\cdot|_{\overline{v}}$ be the unique valuation on $\overline{F}_v$ extending $|\cdot|_v$ on F_v. Let $|\cdot|_{v(\tau)}$ be the valuation on $\overline{F}$ defined by $|a|_{v(\tau)} = |\tau(a)|_{\overline{v}}$. Show that the map $\tau \mapsto |\cdot|_{v(\tau)}$ induces a bijection between
$$\mathrm{Gal}(\overline{F}_v/F_v)\backslash \mathrm{Hom}_F(\overline{F}, \overline{F}_v) \;\leftrightarrow\; \{\text{valuations on } \overline{F} \text{ extending } v \text{ up to equiv.}\}$$

4. (Product formula) Let $|\cdot|_p$ and $|\cdot|_\infty$ be p-adic and arch valuations on $\mathbb{Q}$ (where p is a prime) normalized so that $|p|_p = p^{-1}$ and $|\cdot|_\infty$ is the usual absolute value. Let V be the set of all $|\cdot|_p$ and $|\cdot|_\infty$, which is in bijection with the set of all places of $\mathbb{Q}$. Show that
$$\prod_{v \in V} |a|_v = 1, \quad \forall\, a \in \mathbb{Q}^\times.$$
Moreover, let K be a finite extension of $\mathbb{Q}$ and denote by V_K the set of places of K. Find a suitable normalization of $|\cdot|_w$ for each $w \in V_K$ (within its equivalence class of valuations) such that
$$\prod_{w \in V_K} |a|_w = 1, \quad \forall\, a \in K^\times.$$

   * The following are problems from Serre III.§4.

5. Let A be a Dedekind domain with fraction field K. Let L be a finite separable extension of K and B the integral closure of A in L. Let C be a subring of B, containing A, and having the same field of fractions as B.[1] Show that among all the ideals of B contained in C, there is a largest one, an d that it is the annihilator of the C-module B/C; it is denoted $\mathfrak{f}_{C/B}$ (the "conductor" of B in C). Moreover show that $\mathfrak{f}_{C/B} = \{x \in L : xC^* \subset B^*\}$.

   (Recall that $B^* := \{y \in L : \mathrm{tr}_{L/K}(xy) \in A,\ \forall x \in B\}$ and C^* is defined similarly.)

6. In problem 5, suppose that there exists an ideal $\mathfrak{c}$ of C such that $\mathfrak{c}C^* = C$. Prove that $\mathfrak{f}_{C/B} = \mathfrak{c} \cdot \mathfrak{D}^{-1}_{B/A}$.

[1] For a simple example, take $K = \mathbb{Q}$, $A = \mathbb{Z}$, $L = \mathbb{Q}(\sqrt{D})$ for a square-free D, $B = \mathcal{O}_L$ and $C = \mathbb{Z} + \mathbb{Z}f\sqrt{D}$ for some $f \in \mathbb{Z}_{\ge 1}$.