18.786. (Fall 2011) Homework #5 (due Tue Oct 25)

1. (Neukirch II.1.5) The field $\mathbb{Q}_p$ of p-adic numbers has no field automorphisms except the identity.

2. (Neukirch II.2.6) The fields $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic unless $p = q$.

3. (Neukirch II.3.3) Let k be a field and $K = k(t)$ the function field in one variable. Show that the valuations $v_{\mathfrak{p}}$ associated to the prime ideals $\mathfrak{p} = (p(t))$ of $k[t]$, together with the degree valuation v_∞, are the only valuations of K, up to equivalence.

4. (Neukirch II.4.1) An infinite algebraic extension of a complete field K is never complete.

5. (Neukirch II.4.2) Let $X_0, X_1, ...$ be an infinite sequence of unknowns, p a fixed prime number and

   $$W_n = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n, \quad n \geq 0.$$

   Show that there exist polynomials

   $$S_0, S_1, ...; P_0, P_1, ... \in \mathbb{Z}[X_0, X_1, ...; Y_0, Y_1, ...]$$

   such that

   $$W_n(S_0, S_1, ...) = W_n(X_0, X_1, ...) + W_n(Y_0, Y_1, ...)$$
   $$W_n(P_0, P_1, ...) = W_n(X_0, X_1, ...) \cdot W_n(Y_0, Y_1, ...).$$

6. (Neukirch II.4.3) Let A be a commutative ring. For

   $$a = (a_0, a_1, ...), \quad b = (b_0, b_1, ...), \quad a_i, b_i \in A,$$

   put

   $$a + b = (S_0(a, b), S_1(a, b), ...), \quad a \cdot b = (P_0(a, b), P_1(a, b), ...).$$

   Show that with these operations the vectors $a = (a_0, a_1, ...)$ form a commutative ring $W(A)$ with 1. It is called the **ring of Witt vectors** over A.