18.786. (Fall 2011) Homework # 2 (due Tue Sep 27)

We will do some series of exercises in Neukirch. Problems 1-4 and 5-6 may go together.

1. (Neukirch I.5.2) Show that the convex, centrally symmetric set

$$X = \left\{ (z_\tau) \in K_\mathbb{R} \,\middle|\, \sum_\tau |z_\tau| < t \right\}$$

   has volume $\mathrm{vol}(X) = 2^r \pi^s \frac{t^n}{n!}$. (Here r and s are such that the number field K has r real embeddings and $2s$ complex embeddings.)

2. (Neukirch I.5.3, Minkowski bound) Let K be a number field with $[K : \mathbb{Q}] = n$, and s be as above. Show that in every ideal $\mathfrak{a} \neq 0$ of $\mathcal{O}_K$ there exists an $a \neq 0$ such that

$$|N_{K/\mathbb{Q}}(a)| \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}\mathfrak{N}(\mathfrak{a}).$$

   (Here $\mathfrak{N}(\mathfrak{a})$ denotes the absolute norm $(\mathcal{O}_K : \mathfrak{a})$ of $\mathfrak{a}$.) Hint: Use Exercise I.5.2 to proceed as in Theorem I.5.3, and make use of the inequality between arithmetic and geometric means,

$$\frac{1}{n} \sum_\tau |z_\tau| \geq \left(\prod_\tau |z_\tau| \right)^{1/n}.$$

3. (Neukirch I.6.3) Show that in every ideal class of an algebraic number field K of degree n, there exists an integral ideal $\mathfrak{a}$ such that

$$\mathfrak{N}(\mathfrak{a}) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}.$$

   Hint: Using Exercise I.5.3, proceed as in the proof of Theorem I.6.3.

4. (Neukirch I.6.2) Show that the quadratic fields with discriminant 5, 8, 11, -3, -4, -7, -8, -11 have class number 1.

5. (Neukirch I.6.6) Let $\mathfrak{a}$ be an integral ideal of K and $\mathfrak{a}^m = (a)$. Show that $\mathfrak{a}$ becomes a principal ideal in the field $L = K(\sqrt[m]{a})$, in the sense that $\mathfrak{a}\mathcal{O}_L = (\alpha)$. Clarification of notation: Here $(a) = a\mathcal{O}_K$ and $(\alpha) = \alpha\mathcal{O}_L$ by convention. An integral ideal of K means an ideal of $\mathcal{O}_K$ in the usual sense.

6. (Neukirch I.6.7) Show that, for every number field K, there exists a finite extension L such that every ideal of K becomes a principal ideal.

$*$ If you do not own a copy of Neukirch but want to refer to the relevant part, contact me by email for help.