18.786. (Fall 2011) Homework # 1 (due Tue Sep 20)

1. Let $F = \mathbb{Q}(\sqrt{D})$, where $0 \neq D \in \mathbb{Z}$ is a square-free integer such that $D \neq \pm 1$. According to what we have learned, the ring of integers $\mathcal{O}_F$ is a free $\mathbb{Z}$-module of rank $[F : \mathbb{Q}] = 2$. Exhibit $\mathcal{O}_F$ by writing down a $\mathbb{Z}$-basis for $\mathcal{O}_F$ and checking that it is indeed a $\mathbb{Z}$-basis. Compute d_F, the discriminant of F. (Remark: You may need to do some case-by-case analysis.)

2. Let ζ_n be a primitive n-th root of unity ($n \in \mathbb{Z}_{>2}$) in an algebraic closure of $\mathbb{Q}$, i.e. $(\zeta_n)^n = 1$ and $\zeta_m \neq 1$ for any $1 < m < n$. Let $F = \mathbb{Q}(\zeta_n)$. Then it turns out that $\mathcal{O}_F = \mathbb{Z}[\zeta_n]$ (the subring of F generated by $\mathbb{Z}$ and ζ_n). Compute d_F. (You need not prove that $\mathcal{O}_F = \mathbb{Z}[\zeta_n]$, which will be treated later in the course, but you may give it a try.)

3. Let k be a field. Let A be a k-algebra which is an integral domain. If A is finitely generated as a k-module then show that A is a field. (This fact was used in class.)

4. (Neukirch I.2.7) For any number field K, prove that d_K is either 0 or 1 modulo 4. (Hint[1]: The determinant $\det(\sigma_i \omega_j)$ of an integral basis ω_j is a sum of terms, each prefixed by a positive or a negative sign. Writing P, resp. N, for the sum of the positive, resp. negative terms, one finds $d_K = (P - N)^2 = (P + N)^2 - 4PN$.)

5. (Neukirch I.3.4) Prove that a Dedekind domain with a finite number of prime ideals is a principal ideal domain. (Hint: If $\mathfrak{a} = \mathfrak{p}_1^{\nu_1} \cdots \mathfrak{p}_r^{\nu_r} \neq 0$ is an ideal, then choose elements $\pi_i \in \mathfrak{p}_i \backslash \mathfrak{p}_i^2$ and apply the Chinese remainder theorem for the cosets $\pi_i^{\nu_i} \bmod \mathfrak{p}_i^{\nu_i + 1}$.)

6. (Neukirch I.3.6) Show that every ideal of a Dedekind domain can be generated by two elements. (Hint: It would be useful to show that $A/\mathfrak{a}$ is a PID for an ideal $0 \neq \mathfrak{a}$ of a Dedekind domain A. For this the following hint applies: For $\mathfrak{a} = \mathfrak{p}^n$ the only proper ideals of $A/\mathfrak{a}$ are given by $\mathfrak{p}/\mathfrak{p}^n$, ..., $\mathfrak{p}^{n-1}/\mathfrak{p}^n$. Choose $\pi \in \mathfrak{p} \backslash \mathfrak{p}^2$ and show that $\mathfrak{p}^\nu = (\pi^\nu) + \mathfrak{p}^n$.)

[1] This is copied from Neukirch so as not to give disadvantage to those who do not own a copy. The same principle applies to the hints for other exercise taken from Neukirch.