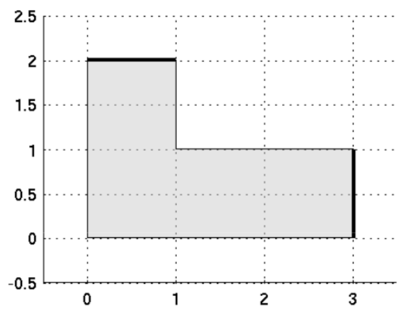
18.086 - Spring 2015, Problem Set 2

March 5, 2016

Out: Mar. 5
Due: Mar. 26

Problem 1 (40 points) Consider the domain Ω (see figure below). At the boundary Γ, we impose Dirichlet boundary conditions at parts Γ_D (thin lines) and Neumann boundary conditions at parts Γ_N (bold lines).



Using finite differences, we would like to approximate the diffusion problem

$$u_t = D\triangle u \quad \text{in } \Omega \tag{1}$$

$$u(0) = 0 \quad \text{in } \Omega \tag{2}$$

$$u = f \quad \text{on } \Gamma_D \tag{3}$$

$$\frac{\partial u}{\partial n} = \frac{\partial f}{\partial n} \quad \text{on } \Gamma_N \tag{4}$$

where $D = 1$ for simplicity and $f(x, y) = x^3y - xy^3 - \frac{1}{2}x^2$.

1. Write a code that numerically solves the above diffusion equation. To this end, introduce a 2D grid of the domain Ω and label each grid position (x, y) by a unique index $j = 1, ..., N$, such that you can write the solution at any time t by a vector $\mathbf{U} = (U_1, U_2, ..., U_N)$. Hint: It's a bit easier if you only include internal grid points in $\mathbf{U}$. Then discretize the Laplacian in 2D by using the 2D centered stencil of Fig. 7.1 in Chapter 7.1 of Strang's book. The stencil will result in a matrix K such that you can write your system as an ODE of the form $\frac{d\mathbf{U}}{dt} = K\mathbf{U}$. Finally, integrate the equation in time using, e.g., forward Euler.

2. Run the code long enough until you approach a steady-state solution (meaning that $\mathbf{U}$ basically doesn't change in time anymore). Plot the steady-state solution $U(x, y)$ on the domain Ω as well as 4 intermediate solutions equally spaced in time.

Problem 2 (20 points) Consider the 1D Burgers equation

$$u_t + uu_x = 0 \tag{5}$$

with initial conditions

$$\begin{aligned} u(x,0) =& u_L \quad \text{for } x < 0 \\ u(x,0) =& u_R \quad \text{for } x > 0 \end{aligned}$$

The correct solution for $u_L > u_R$ is given by the shock wave

$$\begin{aligned} u(x,t) &= u_L \quad \text{for } x < st \\ u(x,t) &= u_R \quad \text{for } x > st \end{aligned} \tag{6}$$

with shock speed $s = (u_L + u_R)/2$.
Confirm that the solution (6) is a weak solution of Burgers' equation, by showing that

$$\int_0^\infty \int_{-\infty}^\infty \left[\phi_t u + \phi_x f(u)\right] dxdt = -\int_{-\infty}^\infty \phi(x,0)u(x,0)dx \tag{7}$$

holds for all compact, smooth test functions $\phi(x,t)$. Here, $f(u) = \frac{u^2}{2}$ is the flux corresponding to Burgers' equation.

Problem 3 (20 points) Consider Burgers' equation

$$u_t + uu_x = 0$$

with a Gaussian initial condition $u_0(x) = \exp(-x^2)$. Choose your computational interval large enough, such that the solution is essentially 0 at the boundaries. Compare the numerical solution obtained by a nonconservative upwind method to the solution obtained by the conservative upwind method

1. up to a time at which the analytical solution is still smooth
2. up to about twice the time after a shock has appeared in the analytical solution.

Hint: You may find the matlab code mit18086_fd_transport_limiter.m from the CSE website useful.
Note: Since we don't know the analytical solution, use numerics to estimate it.