Nonmonotonic effects of parallel sidewalls on Casimir forces between cylinders

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We analyze the Casimir force between two parallel infinite metal cylinders with nearby metal plates using two methods. Surprisingly, the attractive force between cylinders depends nonmonotonically on the separation from the plate(s), and the cylinder-plate force depends nonmonotonically on the separation of the cylinders. These multibody phenomena do not follow from simple two-body force descriptions. We can explain the nonmonotonicity with the screening (enhancement) of the interactions by the fluctuating charges (currents) on the two cylinders and their images on the nearby plate(s).

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Casimir forces arise from quantum vacuum fluctuations, and have been the subject of considerable theoretical and experimental interest [1–4]. We consider here the force between metallic cylinders with one or two parallel metal sidewalls (Fig. 1) using two independent exact computational methods, and find an unusual nonmonotonic dependence of the force on the sidewall separation. These nonmonotonous effects cannot be predicted by commonly used two-body methods, and find an unusual nonmonotonic dependence of the force on the sidewall separation. These nonmonotonous effects cannot be predicted by commonly used two-body Casimir-force estimates, such as the proximity-force approximation (PFA) [5,6] that is based on the parallel-plate limit, or by addition of Casimir-Polder “atomic” interactions (CPI) [6–8].

In previous work, we demonstrated a similar nonmonotonic force between two metal squares in proximity to two parallel metal sidewalls, for either perfect or realistic metals [9]. This work, with perfect-metal cylinders [10], demonstrates that the effect is not limited to squares (i.e., it does not arise from sharp corners or parallel flat surfaces), nor does it require two sidewalls. The nonmonotonicity stems from a competition between forces from transverse electric (TE) and transverse magnetic (TM) field polarizations: In the latter case, the interaction between fluctuating charges on the cylinders is screened by opposing image charges, in the former case it is enhanced by analogous fluctuating image currents. Furthermore, we show that a related nonmonotonic variation arises for the force between the cylinders and a sidewall as a function of separation between the cylinders, a geometry potentially amenable to experiment.

Casimir forces are not two-body interactions: quantum fluctuations in one object induce fluctuations throughout the system, which in turn act back on the first object. However, both the PFA and CPI view Casimir forces as the result of attractive two-body (“pairwise”) interactions. They are reasonable approximations only in certain limits (e.g., low curvature for PFA), and can fail qualitatively as well as quantitatively otherwise. Pairwise estimates fail to account for two important aspects of the Casimir forces in the geometry we consider [11]. First, a monotonic pairwise attractive force clearly cannot give rise to the nonmonotonic effect of the sidewalls. Second, the application of either method here would include two contributions to the force on each cylinder: attraction to the other cylinder and attraction to the sidewall(s). If the latter contribution is restricted to the portion of the sidewall(s) where the other cylinder does not interpose (“line of sight” interactions), the cylinder will experience greater attraction to the opposite side, thereby reducing the net attractive force between the cylinders [11]. In contrast, exact calculations predict a nonmonotonic force that is larger in the limit of close sidewalls than for no sidewalls. These important failures illustrate the need for caution when applying uncontrolled approximations to new geometries even at the qualitative level. (On the other hand, a ray-optics approximation, which incorporates nonadditive many-body ef-

FIG. 1. (Color online) Casimir force per unit length between two cylinders (black) vs the ratio of sidewall separation to cylinder radius $h/R$, at fixed $a/R=2$, normalized by the total PFA force per unit length between two isolated cylinders [$F_{\text{PFA}}=\frac{5}{16}(\hbar c^2/1920)\sqrt{R/a^3}$ [18]]. The solid lines refer to the case with one sidewall, while dashed lines depict the results for two sidewalls, as indicated by the inset. Also shown are the individual TE (red) and TM (blue) forces.

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fects, at least qualitatively predicts these features for the case of two square and/or sidewall [11].}

The insets in Fig. 1 illustrate the three-dimensional geometries that we consider: two infinite, parallel, perfect-metal cylinders of radius $R$ separated by a distance $a$ (center-to-center separation $2R+a$) and oriented along the $z$ axis, with one [Fig. 1(A)] or two [Fig. 1(B)] infinite metal sidewall(s) parallel to the cylinders and separated from both by equal distance $h$ ($h+R$ to the cylinder axes). For perfectly conducting objects with $z$-translational symmetry, the electric ($E$) and magnetic ($H$) fields can be decomposed into TE and TM polarizations, described by scalar fields $\psi$ satisfying Neumann (TE, $\psi = H_z$) and Dirichlet (TM, $\psi = E_z$) boundary conditions at the metallic surfaces [12].

To analyze these geometries, we employ two complementary and exact computational methods, based on path integrals (PI) and the mean stress tensor (ST). The methods are exact as they involve no uncontrolled approximations and can yield arbitrary accuracy given sufficient computational resources. They are complementary in that they have different strengths and weaknesses. The PI method is most informative at large separations where it leads to analytical asymptotic expressions. The ST method, while relatively inefficient for large separations or for the specific geometries where PI is exponentially accurate, is formulated in a generic fashion that allows it to handle arbitrary complex shapes and materials without modification. As both methods are described in detail elsewhere [13,14], we only summarize them briefly here. The present geometry provides an arena where both methods can be applied and compared.

In the PI approach, the Casimir force is calculated via the constrained partition function. The Dirichlet (Neumann) constraints on the TM (TE) fields are imposed by auxiliary fields on the metallic surfaces [15] which can be interpreted as fluctuating charges (currents). The interaction between these charges is related to the free-space Green's functions—the addition of metallic sidewall(s) merely requires using image charges (currents) to enforce the appropriate boundary conditions. The calculations are further simplified by using Euclidean path integrals and the corresponding imaginary-frequency $\omega = iw$ Green's function. In the case of infinite cylinders, these surface fields can be represented in terms of a spectral basis: their Fourier series, leading to Bessel functions, of cylinders, these surface fields can be represented in terms of their Fourier series, leading to Bessel functions.

An intuitive perspective of the effects of the metallic sidewall(s) on the TE and/or TM forces is obtained from the "method of images," whereby the boundary conditions at the plate(s) are enforced by appropriate image sources. For the Dirichlet boundary conditions (TM polarization) the image charges have opposite signs, and the potential due to a charge (more precisely, the Green's function at any imaginary frequency, which determines the Casimir force) is obtained by subtracting the contribution from the opposing image. Any configuration of fluctuating TM charges on one cylinder is thus screened by images, more so as $h$ is decreased, reducing the force on the fluctuating charges of the second cylinder [33]. Since the reduction in force is present for every configuration, it is there also for the average over all configurations, accounting for the variations of the TM curves in Fig. 1. By contrast, the Neumann boundary conditions (TE polarization) require image sources (current loops) of the same sign. The total force between fluctuating currents on the cylinders is now larger and increases as the plate separation $h$ is reduced. (An analogous additive effect occurs for the classical force between current loops near a conducting plane.)

Note, however, that while for each fluctuating source configuration, the effect of images is additive, this is no longer the case for the average over all configurations. More precisely, the effect of an image source on the Green's function...
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The nonmonotonic effect in Fig. 3 is rather small compared to our previous results in Fig. 1. Note that if \( a \) is too large or too small, the degree of nonmonotonicity (defined as the difference between the minimum force and the \( h=0 \) force) decreases. [For small \( a \), the force is accurately described by PFA, while for large \( a \) the TM mode dominates as indicated in Eq. (1).] The separation \( a/R=2 \) from Fig. 1 seems to achieve the largest value of nonmonotonicity.

When the force between the cylinders is not monotonic in \( h \), it also follows that the force between the cylinders and the sidewalls is not monotonic in \( a \). A nonmonotonic force \( F_x \) between the cylinders means that there is a value of \( h \) where \( dF_x/da=0 \). Since the force is the derivative of the energy, \( F_x=-\partial \mathcal{E}/\partial a \), at this point \( \partial^2 \mathcal{E}/\partial a \partial h=0 \). These two derivatives, of course, can be interchanged to obtain \( \partial (\partial \mathcal{E}/\partial h)/\partial a =0 \). But this means that \( dF_x/da=0 \) at the same point, where \( F_x=-\partial \mathcal{E}/\partial h \) is the force between the cylinders and the sidewall. This cylinders-sidewall force is plotted in Fig. 3 as a function of \( a/R \) for various values of \( h/R \). (It is not surprising that the effect of a small cylinder on the force between two bodies is smaller than the effect of an infinite plate. In future work we also show that the cylinder-cylinder force is generally less than the cylinder-plate force for the same cylinder diameter and surface separation [18].)

The advantage of the cylinder-plate force compared to the cylinder-cylinder force is that it seems operationally closer to the sphere-plate geometries that have been realized experimentally. In order to measure the cylinder-cylinder force, one would need to suspend two long cylinders in vacuum nearly parallel to one another. To measure the cylinder-plate force, the cylinders need not be separated by vacuum—we expect that a similar phenomenon will arise if the cylinders are separated by a thin dielectric spacer layer. Unfortunately, the nonmonotonic effect in Fig. 3 is rather small (roughly 0.2%).

### Graphs

**Fig. 2.** (Color online) Casimir force per unit length between two cylinders of fixed radius \( R \) vs the ratio of sidewall separation to cylinder radius \( h/R \) (for one plate), normalized by the total PFA force per unit length between two isolated cylinders. The force is plotted for different cylinder separations of \( a/R=0.2, 0.6, 1.0, 2.0, 3.0, \) and 4.0.

**Fig. 3.** (Color online) Casimir force per unit length between a plate and two cylinders of fixed radius \( R \) vs the ratio of cylinder separation to cylinder radius \( a/R \), normalized by the total PFA force per unit length between a cylinder and a plate. The force is plotted for different plate separations of \( h/R=0.28, 0.6, 1.0, \) and 2.0. Note that the normalization is different from the cylinder-cylinder PFA in the previous figures.
but it may be possible to increase this by further optimization of the geometry. In future calculations, we also hope to determine whether the same phenomenon occurs for two spheres next to a metal plate.

Our aim in this paper has been to establish the existence of nonmonotonic and non-two-body Casimir forces as a matter of principle. If, however, one were to attempt to observe these effects experimentally, one would need to take into account thermal fluctuations, surface roughness, and finite thermal conductivity. This has been undertaken for other geometries \cite{10,19,27}; for a review of the controversy on finite conductivity calculations, see \cite{28}. For the typical length scales studied experimentally the roughness corrections are assumed to be small, while thermal fluctuations ought to be unimportant in the submicron range. Of course, issues of parallel or skewed alignment in experiments are present in a similar manner for other geometries as has been discussed thoroughly by other authors \cite{19,20}.

In previous research, unusual Casimir force phenomena were sought by considering parallel plates with exotic materials: for example, repulsive forces were predicted using magnetic conductors \cite{29}, combinations of different dielectrics \cite{30}, fluids between the plates \cite{31}, and even negative-index media with gain \cite{32}. A different approach is to use ordinary materials with more complicated geometries: as illustrated in this and previous \cite{9} work, surprising nonmonotonic (attractive) effects can arise by considering as few as three objects.

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\bibitem{33} A key fact is that the Green’s functions in Casimir forces are naturally evaluated at imaginary frequencies \cite{13}, which means that they are decaying and not oscillating. If they were oscillating, one could not as easily infer whether opposite-sign image currents add or subtract.
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