Notes on discontinuous $f(x)$ satisfying $f(x+y) = f(x) \cdot f(y)$

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1 Introduction

It is well known that exponential functions $f(x) = e^{kx}$, for any $k \in \mathbb{R}$, are isomorphisms from addition to multiplication, i.e. for all $x, y \in \mathbb{R}$:

$$f(x+y) = f(x) \cdot f(y). \tag{1}$$

In fact, exponentials are the *only* non-zero anywhere-continuous functions over the reals ($\mathbb{R}$) with this property. This is proved below, and is a simple enough result that it has been posed as a homework problem [**?**]. This immediately raises the question, however: *is there a discontinuous function satisfying (1)*? The answer is *yes*, but it is surprisingly non-trivial to prove.

I was initially unable to find any published reference to this fact, although I couldn't believe that it was a new result, so I wrote up the proof below. Inquiries with colleagues in the math department proved fruitless, nor was I able to find the needle in the haystack of real-analysis textbooks in the library. Subsequently, however, my friend Yehuda Avniel, revealing an unexpected background in real analysis, pointed out that the existence of such a function is proved in an exercise of Hewitt and Stromberg [**?**]. It turns out to be quite easy to do once you have proved the existence of a Hamel basis for $\mathbb{R}/\mathbb{Q}$ (a construct I was unfamiliar with). In fact, Hewitt and Stromberg show that it is sufficient to assume that $f(x)$ is merely measurable in order to get exponentials (I sketch the proof below).

Nevertheless, I present my construction of a discontinuous $f(x)$ below, in an elementary tutorial-style fashion, in the hope that it will be useful to a student or two. Note that this is not an *explicit* construction, only a proof that such a function exists; the Hamel basis method of Hewitt and Stromberg is similarly non-constructive. Note also that all of the proofs I know of require the axiom of choice.

2 General properties of $f(x) \neq 0$

Let us begin by proving several useful properties of $f(x)$, only assuming that it is nonzero at some x_0.

- If $f(x_0) \neq 0$ for any x_0, then $f(x) \neq 0$ for all x. *Proof:* $f(x_0) = f(x) \cdot f(x_0 - x) \neq 0$.
- $f(0) = 1$. *Proof*: $f(x+0) = f(x) = f(x) \cdot f(0)$, and $f(x) \neq 0$.
- $f(-x) = f(x)^{-1}$ for all x. *Proof:* $f(-x) \cdot f(x) = f(0) = 1$.
- $f(x) > 0$ for all x. *Proof:* $f(x) = f(x/2)^2 > 0$.
- If $f(x)$ is continuous at $x = y$ for *any* y, then $f(x)$ is continuous at *all* x. Consequently, if $f(x)$ is discontinuous *anywhere*, it is discontinuous *everywhere*. *Proof:* $f(x+\delta) - f(x) = f(x-y) \cdot [f(y+\delta) - f(y)] \to 0$ for $\delta \to 0$ by continuity at y.

3 $f(q)$ for rational q

We can easily show that we must have $f(q) = e^{kq}$ for some $k \in \mathbb{R}$ and non-zero $f(x)$, whenever $q \in \mathbb{Q}$ (q rational). It suffices to show this for positive rational q since $f(-q) = f(q)^{-1} = e^{-kq}$ and $f(0) = 1$ from above.

Proof: Let $q = n/m$ for n and m positive integers. By elementary induction, $f(\frac{n}{m}) = f(\frac{1}{m} + \cdots + \frac{1}{m}) = f(\frac{1}{m})^n$. Therefore, $f(\frac{1}{m})^m = f(1)$ and so $f(\frac{1}{m}) = f(1)^{1/m}$. Thus, we have $f(\frac{n}{m}) = f(1)^{n/m}$. Since $f(1) > 0$ from above, we can write $f(1) = e^k$ for some real $k = \ln f(1)$, and thus $f(q) = e^{kq}$ for all $q \in \mathbb{Q}$.

If we were now to assume that $f(x)$ were continuous, it would follow that $f(x) = e^{kx}$ everywhere, since the closure of $\mathbb{Q}$ is $\mathbb{R}$.

4 Measurable functions

It turns out to be sufficient to assume that $f(x)$ is measurable or Lebesgue integrable, and not identically zero, in order to obtain exponentials from $f(x+y) = f(x)f(y)$. The proof runs as follows. Since $f(x)$ is integrable, we can define $g(x) = \int_0^x f(x')dx'$. Therefore, $g(x+y) - g(x) = \int_x^{x+y} f(x')dx' = \int_0^y f(x'+x)dx' = f(x)g(y)$. Then, if we choose a y such that $g(y) \neq 0$ (which must exist since $f(x)$ is everywhere non-zero, from above), we obtain:

$$
\begin{aligned}
f(x+\delta) - f(\delta) &= \frac{[g(x+\delta+y) - g(x+\delta)] - [g(x+y) - g(x)]}{g(y)} \\
&= \frac{[g(x+y+\delta) - g(x+y)] - [g(x+\delta) - g(x)]}{g(y)} \\
&= \frac{f(x+y)g(\delta) - f(x)g(\delta)}{g(y)} = g(\delta)\frac{f(x+y) - f(x)}{g(y)},
\end{aligned}
$$

and the final expression must go to zero as $\delta \to 0$, since $g(0) = 0$ and $g(x)$ is continuous. Therefore $f(x)$ is continuous, and the result follows from above.

5 A single irrational point

We have now shown that $f(x) = e^{kx}$ for all rational x, and will try to construct a *dis*continuous function at an *ir*rational x. Let us consider a single irrational point $u_1 \in \mathbb{R} - \mathbb{Q}$, and suppose that $f(u_1) = e^{\bar{k}u_1}$ for some real $\bar{k} \neq k$. It then follows that $f(q_1 u_1 + q) = e^{\bar{k}q_1 u_1 + kq}$ for all $q_1, q \in \mathbb{Q}$.

Proof: First, $f(\frac{n}{m}u_1) = f(u_1)^{n/m} = e^{\bar{k}(n/m)u_1}$ from the same induction process as in the previous section, for any rational $q_1 = n/m$. Second, $f(q_1 u_1 + q) = f(q_1 u_1) \cdot f(q) = e^{\bar{k}q_1 u_1 + kq}$.

The consequence of this result is that specifing $f(x)$ for the rationals and a single irrational point u_1 immediately specifies it for another dense countable set $C_1 = \{q_1 u_1 + q \,|\, q_1, q \in \mathbb{Q}, q_1 \neq 0\}$, where C_1 is purely irrational (disjoint from $\mathbb{Q}$).

Similarly, if we now pick a second irrational point $u_2 \in \mathbb{R} - \mathbb{Q} - C_1$ and define $f(u_2) = e^{\bar{k}u_2}$, we must define $f(q_1 u_1 + q_2 u_2 + q) = e^{\bar{k}(q_1 u_1 + q_2 u_2) + kq}$ for all $q_1, q_2, q \in \mathbb{Q}$.

6 A simplistic, incomplete construction

Now, let us give a simplistic, incomplete construction of a discontinuous $f(x)$ satisfying $f(x+y) = f(x) \cdot f(y)$. Although this construction turns out to be unworkable, it illustrates the essential ideas that we will employ in a more complete form below. The construction is as follows:

1. Start by defining $f(q) = e^{kq}$ for some $k \in \mathbb{R}$ and for all $q \in \mathbb{Q}$.

2. Then, define $f(qu_1 + q') = e^{\bar{k}q_1 u_1 + kq}$ for some irrational $u_1 \in \mathbb{R} - \mathbb{Q}$, real $\bar{k} \neq k$, and for all $q_1, q \in \mathbb{Q}$, extending our definition to a set $S_1 = \{q_1 u_1 + q \,|\, q_1, q \in \mathbb{Q}\}$ (with $\mathbb{Q} \subset S_1$).

3. Pick another irrational $u_2 \in \mathbb{R} - S_1$, and define $f(q_1 u_1 + q_2 u_2 + q) = e^{\bar{k}(q_1 u_1 + q_2 u_2) + kq}$ for all $q_1, q_2, q \in \mathbb{Q}$, extending our definition to a set $S_2 = \{q_1 u_1 + q_2 u_2 + q \,|\, q_1, q_2, q \in \mathbb{Q}\}$ (with $\mathbb{Q} \subset S_1 \subset S_2$).

4. Pick another irrational $u_3 \in \mathbb{R} - S_2$ with $f(u_3) = e^{\bar{k}u_3}$, and so on *ad infinitum.*

In this way, we gradually cover more and more of $\mathbb{R}$ with our discontinuous $f(x)$ definition, all the while preserving the property that $f(x+y) = f(x) \cdot f(y)$ for all of the points where $f(x)$ is defined.

The problem with this approach, of course, is that we will never cover all of $\mathbb{R}$ in this way. We are defining $f(x)$ over a countable sequence of countable sets, but the union of such a sequence is only countable and thus has measure zero in $\mathbb{R}$ (despite being dense). To actually cover all of $\mathbb{R}$ by this sort of approach, we must generalize our process to one of transfinite induction over an uncountable set. In particular, the uncountable set in question turns out to be a set of equivalence classes on $\mathbb{R}$.

7 Equivalence classes

The key to defining $f(x)$ seems to be the following equivalence relation on $\mathbb{R}$:

$$x \sim y \Longleftrightarrow x = qy + q' \text{ for some } q, q' \in \mathbb{Q},\ q \neq 0.$$

It is easy to show that this relation satisfies the usual properties ($x \sim x$, $x \sim y \Rightarrow y \sim x$, and $x \sim y, y \sim z \Rightarrow x \sim z$), and therefore partitions $\mathbb{R}$ into a set $\mathscr{C}$ of disjoint equivalence classes C. For each equivalence class C we can pick a single element $u(C) \in C$, and all other elements of that class are then given by $u(C)q + q'$ for $q, q' \in \mathbb{Q}$, $q \neq 0$. Thus every C is countable, and therefore $\mathscr{C}$ must be uncountable. One special equivalence class $C = \mathbb{Q}$ is given by $u(\mathbb{Q}) = 0$.

The significance of these equivalence classes, as explained above, is that once we define $f(q) = e^{kq}$ for $q \in \mathbb{Q}$ then the value of $f(x)$ for all $x \in C$ is determined by picking $f[u(C)]$ for a single $u(C) \in C$. Suppose we define $f[u(C)] = e^{\bar{k} \cdot u(C)}$ for some $\bar{k} \in \mathbb{R}$ and $\bar{k} \neq k$. (As notational shorthand, we will denote $u(C_n)$ by u_n.) Then for any $x_n = q_n u_n + q'_n \in C_n$ we must have $f(x_n) = e^{\bar{k} q_n u_n + k q'_n}$.

However, we cannot pick $u(C)$ for the different equivalent classes independently, because of what happens when we add numbers from two equivalence classes. First, realize:

- Given $x_1 \in C_1$ and $x_2 \in C_2$ for $C_1 \neq C_2$ and $C_{1,2} \neq \mathbb{Q}$, it follows that $x_1 + x_2 = x_3 \in C_3$ for $C_3 \neq C_{1,2}$, $C_3 \neq \mathbb{Q}$. *Proof:* If $C_3 = C_1$ then $x_3 \sim x_1$ and thus $x_2 = (q-1)x_1 + q'$: if $q = 1$ then $x_2 \sim q'$ and $C_2 = \mathbb{Q}$, while if $q \neq 1$ then $x_2 \sim x_1$ and $C_1 = C_2$. Thus, $C_3 \neq C_{1,2}$. If $C_3 = \mathbb{Q}$ then $x_1 = -x_2 + q$ and $x_1 \sim x_2$ ($C_1 = C_2$).

We thus have $x_1 + x_2 = (q_1 u_1 + q'_1) + (q_2 u_2 + q'_2) = x_3 = q_3 u_3 + q'_3$ for some $q_{1,2,3}, q'_{1,2,3} \in \mathbb{Q}$, $q_{1,2,3} \neq 0$, and $u_1 \neq u_2 \neq u_3$. We must have $f(x_1 + x_2) = e^{\bar{k}(q_1 u_1 + q_2 u_2) + k(q'_1 + q'_2)} = f(x_3) = e^{\bar{k} q_3 u_3 + k q'_3}$. This is only true, however, if $q'_1 + q'_2 = q'_3$, which implies

$$q_1 u_1 + q_2 u_2 = q_3 u_3$$

for some $q_3 \in \mathbb{Q}$. That means we cannot pick the $u(C)$'s independently: they must be defined inductively to satisfy this algebraic relation for some q_3.

Before we do so, we should first check whether we have run into something obviously impossible. Can we have $x_3 = q_1 u_1 + q_2 u_2 = q_3 u_3 \sim \bar{x}_3 = \bar{q}_1 u_1 + \bar{q}_2 u_2 = \bar{q}_3 u_3 + \bar{q}'_3$ for some $q_{1,2,3}, \bar{q}_{1,2,3}, \bar{q}'_3 \in \mathbb{Q}$ and $\bar{q}'_3 \neq 0$? No. *Proof:* $\bar{x}_3 - \frac{\bar{q}_3}{q_3} x_3 = \bar{q}'_3$, but this means $q u_1 + q' u_2 = \bar{q}'_3$ for rational $q = \bar{q}_1 - \frac{\bar{q}_3}{q_3} q_1$ and $q' = \bar{q}_2 - \frac{\bar{q}_3}{q_3} q_2$. If $q \neq 0$ or $q' \neq 0$ then $u_1 \sim u_2$, contradicting our assumption that $C_1 \neq C_2$. If $q = q' = 0$ then $\bar{q}'_3 = 0$ and all is well.

8 Transfinite induction

We will proceed to define our $u(C)$ by transfinite induction on $\mathscr{C}$. First, we must well-order $\mathscr{C}$, by invoking the well-ordering theorem on $\mathscr{C} - \{\mathbb{Q}\}$ to choose some well-order relation "$<$" on equivalence classes, and then put $\mathbb{Q}$ first by defining $\mathbb{Q} < C$ for any $C \neq \mathbb{Q}$. (Recall that a well-ordering is one such that every non-empty set has a least element. Since $\mathscr{C}$ is uncountable, the well-ordering theorem requires the axiom of choice.) Then, we will construct $u(C)$ to satisfy the following property by induction:

- Let $\mathscr{C}_0 = \{C \,|\, \mathbb{Q} < C < C_0\}$ for some $C_0 \in \mathscr{C}$. For all finite series $x = \sum_n q_n u_n$ with distinct $u_n = u(C_n)$, $C_n \in \mathscr{C}_0$, and some $q_n \in \mathbb{Q}$, then whenever $x \in C \in \mathscr{C}_0$ we require $x = q \cdot u(C)$ for some $q \in \mathbb{Q}$.

That is, we assume that the above property is true for all $C < C_0$, and then choose $u_0 = u(C_0)$ so that it still holds when we include C_0 (i.e. for $\mathscr{C}_1 = \mathscr{C}_0 \cup \{C_0\}$). In particular, there are two cases: (i) If $\sum_n q_n u_n \notin C_0$ for any q_n or u_n with $C_n \in \mathscr{C}_0$, then we choose u_0 to be any arbitrary element of C_0. (ii) Otherwise, we pick $u_0 = \sum_n q_n u_n$ for any arbitrary series $\sum_n q_n u_n \in C_0$. Then the desired property above follows: If we have a $\sum_n q'_n u'_n = qu_0 + q' \in C_0$ ($n \neq 0$), then by substituting u_0 and moving it to the left we obtain a sum of the form $\sum_n q''_n u''_n = q'$, which is only possible if $q' = 0$ (if any $q''_n \neq 0$, then we will obtain $u_n \sim u_m$ for some $m \neq n$, or otherwise $u_n \in \mathbb{Q}$), similar to the proof at the end of the previous section. On the other hand, if we have $x = q_0 u_0 + \sum_n q'_n u'_n \in C \in \mathscr{C}_0$, then $x = \sum_n q''_n u''_n$ and thus $x = q \cdot u(C)$ by induction. Note that if $q_0 \neq 0$ then $x \in C$ implies that $\sum_n q'_n u'_n - qu(C) \in C_0$, so we are in case (ii) above.

The base case, for $\mathscr{C}_0$ the empty set, is trivial. We define $u(\mathbb{Q}) = 0$.

9 A discontinuous $f(x)$

Now that we have defined $u(C)$ as above, defining the discontinuous $f(x)$ is easy. Every $x \in \mathbb{R}$ is a member of some equivalence class C, and thus $x = qu(C) + q'$ for some $q, q' \in \mathbb{Q}$, $q \neq 0$. Then, $f(x) = e^{\bar{k}qu(C)+kq}$ for some fixed real numbers $\bar{k} \neq k$. This is discontinuous since $f(q) = e^{kq}$ but $f(x) \neq e^{kx}$ for irrational x.

Let us review why this satisfies $f(x_1 + x_2) = f(x_1) \cdot f(x_2)$ for any $x_1, x_2 \in \mathbb{R}$, where $x_1 = q_1 u_1 + q'_1$ and $x_2 = q_2 u_2 + q'_2$ with $u_1 = u(C_1)$ and $u_2 = u(C_2)$ for $x_1 \in C_1$ and $x_2 \in C_2$. If $C_1 = C_2$ or $C_2 = \mathbb{Q}$, then $f(x_1 + x_2) = e^{\bar{k}(q_1 u_1 + q_2 u_2) + k(q'_1 + q'_2)}$ as desired. Otherwise, $x_1 + x_2 \in C_3 \neq C_{1,2}$, and also $q_1 u_1 + q_2 u_2 \in C_3$. By our construction of $u(C)$, however, $u_3 = u(C_3)$ must then satisfy the property $q_1 u_1 + q_2 u_2 = q_3 u_3$ for some $q_3 \in \mathbb{Q}$. Therefore, $x_1 + x_2 = q_3 u_3 + (q'_1 + q'_2)$ and $f(x_1 + x_2) = e^{\bar{k}q_3 u_3 + k(q'_1 + q'_2)} = e^{\bar{k}(q_1 u_1 + q_2 u_2) + k(q'_1 + q'_2)} = f(x_1) \cdot f(x_2)$.

References