

Saddle-point integration of C_∞ “bump” functions

Steven G. Johnson

June 26, 2007

1 Background

In our work involving optimization of slow “taper” transitions for waveguide couplers, and also for our work on exponential absorbing boundary layers in wave equations, we encountered a problem that could be expressed in terms of the Fourier transform of smooth functions with compact support. In particular, we needed to analyze the asymptotic rate of decay of the Fourier coefficients. This was straightforward for functions with simple discontinuities (via integration by parts) or with simple poles (via contour integration). However, it turned out that some of the most interesting functions were C_∞ (infinitely differentiable) functions with compact support, and these functions have essential singularities that resist the previous methods. In this note, we describe how the asymptotic Fourier transforms of such functions can be analyzed with the help of saddle-point integration [1].¹

In particular, we will look at C_∞ functions on \mathbb{R} with compact support $[-1, 1]$. The canonical example of such a function is the symmetric “bump:”

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} .$$

The actual functions we are interested are more complicated than this (e.g., $f(x)$ multiplied by some analytic function), but their analysis is similar to that of $f(x)$: the key point is that we have essential singularities at $x = -1$ and $x = +1$, and these singularities determine the asymptotic behavior of the Fourier transform and similar integrals.

We should also note that the space of C_∞ functions with compact support plays an important role in the theory of generalized functions (distributions), where they are the “test functions” that are the domain of the distributions. In this context, it has been proven that the Fourier transform of any such test function is an entire function (analytic over the whole complex plane) and

¹We are indebted to our colleagues, Prof. Martin Bazant and especially Prof. Hung Cheng at MIT, for their helpful suggestions in this matter.

diverges at most exponentially fast off the real axis [2]. From the fact that the functions are infinitely differentiable, it also immediately follows that their Fourier transforms go to zero along the real axis faster than the inverse of any polynomial [3]. Here, however, we want to know precisely how fast the Fourier transform decays, and with what coefficient.

2 Saddle-point Fourier integration of $f(x)$

We wish to compute the asymptotic behavior of the Fourier transform

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} dx = 2 \operatorname{Re} \int_0^1 e^{ikx - \frac{1}{1-x^2}} dx$$

for $|\operatorname{Re} k| \gg 1$. (Without loss of generality, we can restrict ourselves to real $k \geq 0$.) To do this, we will employ a saddle-point integration: we will look for the x at which the exponent is stationary, and expand the exponent approximately around this point. For large k , this saddle point of the stationary exponent will dominate the integral, and this will give us the asymptotic behavior. It will turn out that the saddle point occurs for a complex x , however, so we will need to deform the integration contour within the complex plane to exploit this approach.

It is clear (and this will also be justified *a posteriori*) that the biggest contributions must come near the singular point $x = 1$. Exactly *at* the singular point, however, the integrand is zero, so (perhaps) contrary to our intuition the endpoints *per se* are not important. Because we are worrying about points near the endpoint, however, it is convenient to change variables $t = 1 - x$ and write

$$F(k) = 2 \operatorname{Re} \int_0^1 e^{g(t)} dt$$

with

$$g(t) = ik - ikt - \frac{1}{(2-t)t} \approx ik - ikt - \frac{1}{2t} - \frac{1}{4} + O(t),$$

where the last approximation is for $|t| \ll 1$, which is valid (it turns out) at our saddle point. The saddle point is the $t = t_0$ where $g'(t) = 0 = -ik + 1/2t^2$, which by inspection is² $t_0 = \sqrt{1/2ik}$. Note that for large $k \gg 1$ we obtain $|t_0| \ll 1$, justifying our approximation above. Now, we write

$$\begin{aligned} g(t) &\approx g(t_0) + \frac{g''(t_0)}{2}(t - t_0)^2 \\ &= ik - i\sqrt{\frac{-ik}{2}} - \sqrt{\frac{ik}{2}} - \frac{1}{4} + \frac{-(\sqrt{2ik})^3}{2} \left(t - \sqrt{\frac{1}{2ik}} \right)^2 \\ &= ik - \frac{1}{4} - \sqrt{2ik} + \sqrt{-2ik}^{3/2} \left(t - \sqrt{\frac{1}{2ik}} \right)^2. \end{aligned}$$

² g' also vanishes at $t = -\sqrt{1/2ik}$, but we cannot deform our integration contour to a point with negative $\operatorname{Re} t$: for $\operatorname{Re} t < 0$, our integrand $\sim e^{-1/2t}$ blows up.

To actually do this integral, we need to deform our integration contour in the complex plane to lie along a line $t = u/\sqrt{i}$ for real u near $u = 0$, so that the saddle point $t = t_0$ lies along our integration path (at $u = \sqrt{1/2k}$).³ (This deformation is not a problem since we don't have any singularities except at $t = 0$ and $t = 2$, and the integrand vanishes as $t \rightarrow 0$ for $\text{Re } t > 0$.) In this case, our integral becomes (approximately) a Gaussian integral, since:

$$g(u/\sqrt{i}) = ik - \frac{1}{4} - \sqrt{2ik} - \sqrt{2ik}^{3/2} \left(u - \sqrt{\frac{1}{2k}} \right)^2.$$

Recall that the integral of $\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\pi/a}$ as long as $\text{Re } a > 0$, which is true here. Note also that, in the limit as k becomes large, the integrand becomes zero except close to $u = \sqrt{1/2k}$, so we can neglect the rest of the contour and treat the integral over u as going from $-\infty$ to ∞ . (Thankfully, the width of the Gaussian $\Delta u \sim k^{-3/4}$ goes to zero faster than the location of the maximum $u_0 \sim k^{-1/2}$, so we don't have to worry about the $u = 0$ origin.) Also note that the change of variables from t to u gives us a $\sqrt{-i}$ Jacobian factor. Thus, when all is said and done, we obtain the exact asymptotic form of the Fourier integral for $k \gg 1$:

$$F(k) \approx 2 \text{Re} \left[\sqrt{\frac{-i\pi}{\sqrt{2ik}^{3/2}}} e^{ik - \frac{1}{4} - \sqrt{2ik}} \right].$$

Since $\sqrt{2i} = 1 + i$, this means that the Fourier coefficients decay proportional to $k^{-3/4} e^{-\sqrt{k}}$, which is consistent with the expected faster-than-polynomial decay.

To check this result, we can compare the above formula with an exact numerical evaluation of the Fourier integral. Numerical integration was performed using adaptive Clenshaw-Curtis quadrature, specialized for a $\cos(kx)$ oscillatory weight function, from the GNU Scientific Library (adapted from QUADPACK). The resulting $|F(k)|$ for $0 \leq k \leq 150$ is plotted in figure 1. As can be seen from the figure, the asymptotic approximation is an excellent match for the exact result, with errors under 10% for frequencies k as small as 4.

3 Generalized bumps

As a further warm-up for the more complicated problems we'll need to solve later, let's look at the asymptotic Fourier transform of a generalization of our bump function $f(x)$:

$$f_{\alpha,\beta}(x) = \begin{cases} e^{-\frac{\beta}{(1-x^2)^{\alpha-1}}} & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases},$$

³Although $t = u/\sqrt{i}$ is not the path of *steepest* descent around t_0 , as would be prescribed by the usual saddle-point method (a.k.a. the method of "steepest descent"), it is at least a path of descent (the integrand is decaying along that path) [1]. This is sufficient for us to apply our Gaussian integral approximation.

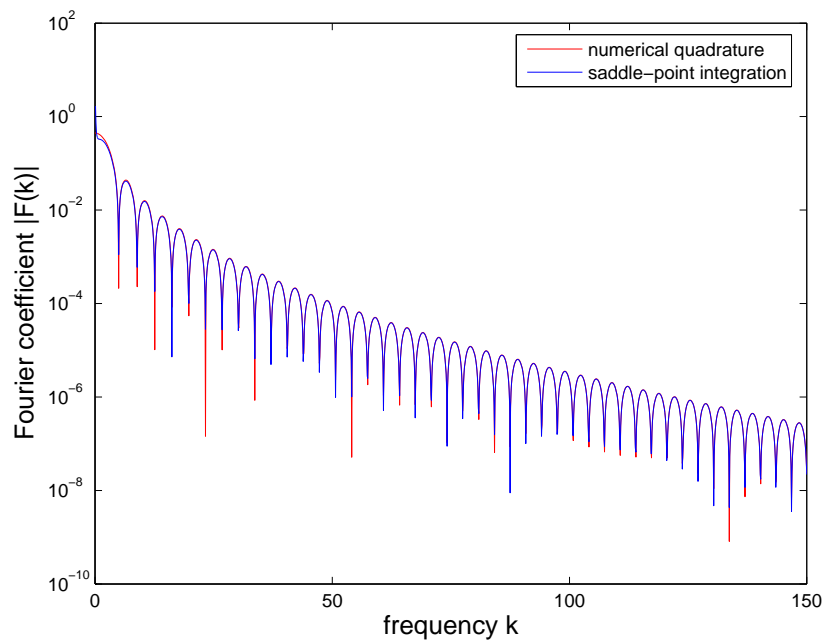


Figure 1: “Exact” numerical quadrature of $|F(k)|$ (red line) and asymptotic saddle-point integration (blue line) for the Fourier transform of the C_∞ bump function $f(x)$. (They match so well that the curves are difficult to distinguish.)

so that $f(x) = f_{2,1}$. Again, we look at the Fourier transform with changed variables $x = 1 - t$, i.e. $\int_0^1 f_{\alpha,\beta}(1-t)e^{ik-ikt}dt$, and expand the exponent $g(t)$ around $t = 0$:

$$g(t) = ik - ikt - \frac{\beta}{(2-t)^{\alpha-1}t^{\alpha-1}} \approx ik - ikt - \frac{\beta}{(2t)^{\alpha-1}} + O(t^{2-\alpha}).$$

Our derivative is then $g'(t) = 0 \approx -ik + \beta(\alpha-1)/2^{\alpha-1}t^\alpha$ at

$$t_0 = [2^{1-\alpha}\beta(\alpha-1)/ik]^{1/\alpha}.$$

That is, $t_0 \sim k^{-1/\alpha}$, and therefore $g(t_0) \sim k^{(\alpha-1)/\alpha}$ asymptotically.

The second derivative is $g''(t_0) = -2^{1-\alpha}\beta\alpha(\alpha-1)/t_0^{\alpha+1} = -i^{(\alpha+1)/\alpha}2Ak^{(\alpha+1)/\alpha}$ where

$$A = \alpha[2\beta(\alpha-1)]^{-1/\alpha}.$$

Again, we'll choose a contour $t = u/i^{1/\alpha}$, in which case we find:

$$g(t) \approx g(t_0) - i^{(\alpha-1)/\alpha}Ak^{(\alpha+1)/\alpha}(u - u_0)^2,$$

which is a path of descent so we can perform a Gaussian integral. The final answer for the integral, including the Jacobian factor for $dt = du/i^{1/\alpha}$, is then

$$F(k) \approx 2 \operatorname{Re} \left[\sqrt{\frac{\pi}{(ik)^{(\alpha+1)/\alpha}A}} e^{ik-ikt_0 - \frac{\beta}{[(2-t_0)t_0]^{\alpha-1}}} \right],$$

with t_0 and A given above. Notice that for our exponent $g(t_0)$ we use the exact form of $g(t)$ and not its approximation for small t . The reason is that, for $\alpha > 2$, terms of $O(t^{2-\alpha})$ do not go to zero, and therefore make a non-negligible multiplicative contribution to the amplitude $F(k)$ even though they do not affect the saddle-point integration.

Numerical tests seem to confirm the accuracy of this $F(k)$ formula, although for $\alpha > 3$ we start to have more difficulty with the numerical quadrature. As one might have expected, increasing either α or β makes $F(k)$ decay more rapidly. To perform such comparisons more carefully, we should typically normalize by $\int_0^1 f_{\alpha,\beta}(x)dx$ or similar, to ignore effects due simply to the fact that the integrand is getting smaller overall.

References

- [1] H. Cheng, *Advanced Analytic Methods in Applied Mathematics, Science, and Engineering*. Boston: LuBan Press, 2006.
- [2] I. M. Gelfand and G. E. Shilov, *Generalized Functions*. New York: Academic Press, 1964.
- [3] K. O. Mead and L. M. Delves, "On the convergence rate of generalized Fourier expansions," *IMA J. Appl. Math.*, vol. 12, no. 3, pp. 247-259, 1973.