18.369 Problem Set 1

Due Friday, 14 February 2014.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of operators $\hat{O}$ by: $\langle H_1, \hat{O}H_2\rangle = \langle \hat{O}^\dagger H_1, H_2\rangle$ for all H_1 and H_2 in the vector space. Show that for a *finite-dimensional* Hilbert space, where H is a column vector h_n ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}, H^{(2)}\rangle$ is the ordinary conjugated dot product $\sum_n h_n^{(1)*} h_n^{(2)}$, the above adjoint definition corresponds to the conjugate-transpose for matrices. (Thus, as claimed in class, "swapping rows and columns" is the *consequence* of the "real" definition of transposition/adjoints, not the source.)

In the subsequent parts of this problem, you may *not* assume that $\hat{O}$ is finite-dimensional (nor may you assume any specific formula for the inner product).

(b) Show that if $\hat{O}$ is simply a number o, then $\hat{O}^\dagger = o^*$. (This is *not* the same as the previous question, since $\hat{O}$ here can act on infinite-dimensional spaces.)

Recall that the key properties of any inner product are $\langle u, v\rangle = \langle v, u\rangle^*$, $\langle u, \alpha v + \beta w\rangle = \alpha\langle u, v\rangle + \beta\langle u, w\rangle$ (for arbitrary complex scalars α, β), and $\|u\|^2 = \langle u, u\rangle \geq 0$ ($= 0$ if and only if[1] $u = 0$).

(c) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called **unitary**. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues u of a unitary operator have unit magnitude ($|u| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(d) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

[1] Technically, we mean $u = 0$ "almost everywhere" (e.g. excluding isolated points).

Problem 2: Maxwell eigenproblems

(a) In class, we eliminated $\mathbf{E}$ from Maxwell's equations to get an eigenproblem in $\mathbf{H}$ alone, of the form $\hat{\Theta}\mathbf{H}(\mathbf{x}) = \frac{\omega^2}{c^2}\mathbf{H}(\mathbf{x})$. Show that if you instead eliminate $\mathbf{H}$, you *cannot* get a Hermitian eigenproblem in $\mathbf{E}$ except for the trivial case $\varepsilon =$ constant. Instead, show that you get a *generalized Hermitian eigenproblem*: an equation of the form $\hat{A}\mathbf{E}(\mathbf{x}) = \frac{\omega^2}{c^2}\hat{B}\mathbf{E}(\mathbf{x})$, where *both* $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For *any* generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle \mathbf{E}, \hat{B}\mathbf{E}\rangle > 0$ for all $\mathbf{E}(\mathbf{x}) \neq 0$), show that the eigenvalues (i.e., the solutions of $\hat{A}\mathbf{E} = \lambda\hat{B}\mathbf{E}$) are real and that different eigenfunctions $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $\mathbf{E}$ eigenproblem above was indeed positive definite.

(c) Alternatively, show that $\hat{B}^{-1}\hat{A}$ is Hermitian under a modified inner product $\langle \mathbf{E}, \mathbf{E}'\rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}'\rangle$ for Hermitian $\hat{A}$ and $\hat{B}$ and positive-definite $\hat{B}$ with respect to the original $\langle \mathbf{E}, \mathbf{E}'\rangle$ inner product; the results from the previous part then follow.

(d) Show that *both* the $\mathbf{E}$ and $\mathbf{H}$ formulations lead to generalized Hermitian eigenproblems with real ω if we allow magnetic materials $\mu(\mathbf{x}) \neq 1$ (but require μ real, positive, and independent of $\mathbf{H}$ or ω).

(e) μ and ε are only ordinary numbers for *isotropic* media. More generally, they are 3×3 matrices (technically, rank 2 tensors)—thus, in an *anisotropic medium,* by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in $\mathbf{E}$ (or $\mathbf{H}$) with real eigenfrequencies ω.