18.369 Midterm Exam (Spring 2014)

You have two hours.

Problem 1: Irreps (50 pts)

Suppose we are in 2d (xy plane), working with the TM polarization ($\mathbf{E}$ out of plane), and have a periodic (period a) surface shown in Fig 1(left). Above the surface is a time-harmonic point source $\mathbf{J} = \delta(x)\delta(y)e^{-i\omega t}\hat{z}$ (choosing the origin to be the location of the point source, for convenience). As you saw in homework, you can define a frequency-domain problem $(\nabla\times\nabla\times - \omega^2\varepsilon)\mathbf{E} = i\omega\mathbf{J}$ (setting $\mu_0 = \varepsilon_0 = 1$ for convenience) for the time-harmonic fields in response to this current.

In this problem, you will explain how to take advantage of the fact that the structure (but not the source or fields!) is periodic, by reducing it to a set of problems of the form shown in Fig. 1(right): solving for the fields of the *same* point source $\mathbf{J}$, but in a *single unit cell* of the structure with *periodic boundary conditions* on the fields.

(a) Use the identity:
$$\delta(x) = \frac{a}{2\pi}\int_0^{2\pi/a}\left[\sum_{n=-\infty}^{\infty}\delta(x-na)e^{ikna}\right]dk$$
to write $\mathbf{J}$ in terms of a superposition of Bloch-periodic sources. What can you say about the resulting $\mathbf{E}$ fields from each term in this superposition?

(b) Write down a linear equation of the form $\hat{A}u = b$, analogous to $(\nabla\times\nabla\times - \omega^2\varepsilon)\mathbf{E} = i\omega\mathbf{J}$, but with *periodic* boundary conditions, whose solution yields the fields from each term in the superposition of the previous part.

(c) Write your total field $\mathbf{E}$ (the solution of the original problem, a single point source) in terms of the fields in the previous part.

(d) Suppose that we want to compute the radiated power P (per unit z) from $\mathbf{J}$ by integrating the Poynting flux through a plane above the current ($y = y_0 > 0$):
$$P = \frac{1}{2}\int_{-\infty}^{\infty}\hat{y}\cdot\mathrm{Re}\left[\mathbf{E}^*(x,y_0)\times\mathbf{H}(x,y_0)\right]dx.$$
Show that $P = \frac{a}{2\pi}\int_0^{2\pi/a}P_k dk$, a simple integral of powers P_k computed *separately* for each periodic subproblem above. (Hint: orthogonality of partner functions.)

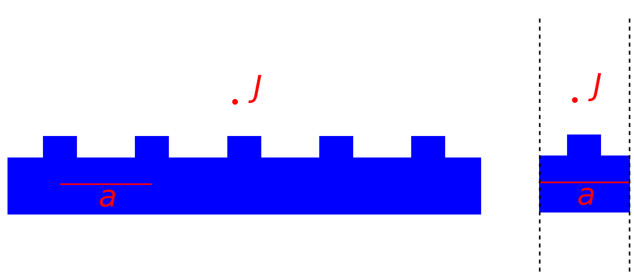


Figure 1: Schematic for problem 1. *Left*: a time-harmonic point source $\mathbf{J}$ above a periodic surface. *Right:* the problem can be reduced to solving a set of problems with point sources in a single unit cell, with periodic boundary conditions on the fields.

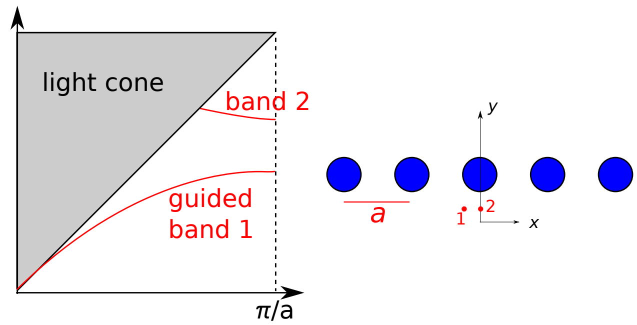


Figure 2: Sketch of TM band diagram of periodic (period a) sequence of dielectric ($\varepsilon = 12$) cylinders in air ($\varepsilon = 1$).

Problem 2: LDOS (50 pts)

Figure 2 shows a qualitative sketch of the typical TM band diagram of a periodic sequence of dielectric ($\varepsilon > 1$) cylinders in air ($\varepsilon = 1$), including the light cone and two guided modes.

(a) Sketch contour plots (label positive and negative) of the E_z fields of the two guided bands at $k = \pi/a$.

(b) From class, the LDOS (for the TM polarization) is $\text{LDOS}(\mathbf{x}, \omega) = \sum_n \varepsilon(\mathbf{x}) |E_z^{(n)}(\mathbf{x})|^2 \delta(\omega - \omega_n)$ for a Hermitian eigenproblem in a finite box (so that we have a discrete basis of modes n, normalized to $\int \varepsilon |\mathbf{E}|^2 = 1$). In class, we also derived that the DOS per-period in a 1d-periodic system is an integral over the bands: $\frac{a}{2\pi} \int_0^{2\pi/a} dk \sum_n \delta(\omega - \omega_n(k))$; the relevant notes are attached in Fig. 3. Show that the LDOS can also be written in terms of a similar integral $\int_0^{2\pi/a} dk$ over the Bloch modes $\mathbf{E}_k(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$, assuming that the Bloch modes are normalized so that $\int \varepsilon |\mathbf{E}|^2 = 1$ for an integral over the unit cell.

(c) Sketch the LDOS as a function of ω for a $\hat{z}$-directed point source at the point 1 in Fig. 2, a point (x_0, y_0) where x_0 is off-center in the unit cell. Explain how this LDOS plot changes if you change y_0. Hint: Fig. 4 includes a copy of some relevant formulas from the class notes on DOS and LDOS.

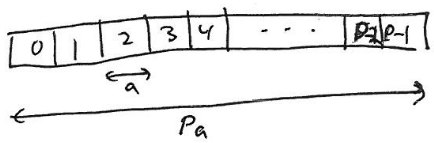
(d) How does your answer from the previous part change if you look instead at the LDOS at the point 2 in Fig. 2, which differs in that it is a point x_0 exactly on the mid-plane of one of the cylinders? (Hint: when the LDOS is expressed as an integral over k as in part (b), how will the dependence upon k change at point 2?)

* Simplest case: assume finite domain ("universe lives in a box"),
Hermitian $\Longrightarrow$ discrete, real eigenfrequencies ω_n

$$\Rightarrow \quad DOS(\omega) = \sum_n \delta(\omega - \omega_n) \qquad \text{(one "spike" per mode)}$$

$$\Longleftrightarrow \quad \int_{\omega_a}^{\omega_b} DOS(\omega)\, d\omega = \#\text{ eigenvalues in } [\omega_a, \omega_b]$$

Next step up: supercell of P unit cells of size $\underline{a}$ (in 1d)



with periodic boundaries ($x \Leftrightarrow x + Pa$)

if Bloch modes of $\underline{\infty}$ periodic system are $H_k(x) e^{ikx}$,

eigenfunctions of P-periodic system are $H_k e^{ikx}$ for $e^{ik(Pa)} = 1$

$\Longleftrightarrow k = \frac{2\pi}{Pa} p$ for $p = 0, 1, \ldots P$

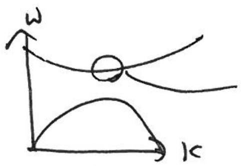
(note: $p + P \Rightarrow k + \frac{2\pi}{a}$ = same sol as k)

$$\Rightarrow \underline{\text{per-period}}\ DOS = DOS(\omega)/P$$

$$= \frac{1}{\cancel{P}} \sum_{p=0}^{P-1} \sum_n \delta\left(\omega - \omega_n\left(\frac{2\pi p}{Pa}\right)\right) \cdot \underbrace{\frac{2\pi}{Pa}}_{\Delta k} \cdot \frac{\cancel{P} a}{2\pi} \xrightarrow[P\to\infty]{} \boxed{\frac{a}{2\pi} \int_0^{2\pi/a} dk \sum_n \delta(\omega - \omega_n(k))}$$

Figure 3: Notes on the per-period DOS, from class.

* Van Hove singularity: 1d example



band edge $\omega \approx \omega_0 + \alpha\left(k - \frac{\pi}{a}\right)^2$

$$DOS(\omega) \approx \int dk\ \delta\left(\omega - \omega_0 - \alpha\left(k - \frac{\pi}{a}\right)^2\right)$$

$$= \int dk\ \frac{\delta\left(k - \frac{\pi}{a} \pm \sqrt{\frac{\omega - \omega_0}{\alpha}}\right)}{2\alpha\left(k - \pi/a\right)} \sim \underline{\underline{\frac{1}{\sqrt{\omega - \omega_0}}}}$$

$\left(k - \frac{\pi}{a}\right)^2 = \frac{\omega - \omega_0}{\alpha}$

$k = \frac{\pi}{a} \pm \sqrt{\frac{\omega - \omega_0}{\alpha}}$

Figure 4: A copy of some relevant formulas from the class notes on DOS and LDOS.