18.369 Problem Set 1

Due Friday, 17 February 2012.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of operators $\hat{O}$ by: $\langle H_1, \hat{O}H_2\rangle = \langle \hat{O}^\dagger H_1, H_2\rangle$ for all H_1 and H_2 in the vector space. Show that for a *finite-dimensional* Hilbert space, where H is a column vector h_n ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}, H^{(2)}\rangle$ is the ordinary conjugated dot product $\sum_n h_n^{(1)*} h_n^{(2)}$, the above adjoint definition corresponds to the conjugate-transpose for matrices. (Thus, as claimed in class, "swapping rows and columns" is the *consequence* of the "real" definition of transposition/adjoints, not the source.)

In the subsequent parts of this problem, you may *not* assume that $\hat{O}$ is finite-dimensional (nor may you assume any specific formula for the inner product).

(b) Show that if $\hat{O}$ is simply a number o, then $\hat{O}^\dagger = o^*$. (This is *not* the same as the previous question, since $\hat{O}$ here can act on infinite-dimensional spaces.)

(c) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called **unitary**. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues u of a unitary operator have unit magnitude ($|u| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(d) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

Problem 2: Completeness

(a) Prove that the eigenvectors H_n of a *finite-dimensional* Hermitian operator $\hat{O}$ (a $d \times d$ matrix) are *complete*: that is, that any d-dimensional vector can be expanded as a sum $\sum_n c_n H_n$ in the eigenvectors H_n with some coefficients c_n. It is sufficient to show that there are d linearly independent eigenvectors H_n. For example, you can follow these steps:

   (i) Show that every $d \times d$ Hermitian matrix O has at least one nonzero eigenvector H_1 [... use the fundamental theorem of algebra: every polynomial with degree > 0 has at least one (possibly complex) root].

   (ii) Show that the space of $V_1 = \{H \mid \langle H, H_1\rangle = 0\}$ orthogonal to H_1 is preserved (transformed into itself or a subset of itself) by $\hat{O}$. From this, show that we can form a $(d-1) \times (d-1)$ Hermitian matrix whose eigenvectors (if any) give (via a similarity transformation) the remaining (if any) eigenvectors of $\hat{O}$.

   (iii) By induction, form an orthonormal basis of d eigenvectors for the d-dimensional space.

(b) Completeness is not automatic for eigenvectors in general. Give an example of a non-singular *non-Hermitian* operator whose eigenvectors are *not* complete. (A 2×2 matrix is fine. This case is also called "defective.")

(c) Completeness of the eigenfunctions is not automatic for Hermitian operators on infinite-dimensional spaces either; they need to have some additional properties (e.g. "compactness") for this to be true. However, it is true of most operators that we encounter in physical problems. If a particular operator did *not* have a complete basis of eigenfunctions, what would this mean about our ability to simulate the solutions on a computer in a finite computational box (where, when you discretize the problem, it turns approximately into a finite-dimensional problem)? No rigorous arguments required here, just your thoughts.

Problem 3: Maxwell eigenproblems

(a) In class, we eliminated $\mathbf{E}$ from Maxwell's equations to get an eigenproblem in $\mathbf{H}$ alone, of the form $\hat{\Theta}\mathbf{H}(\mathbf{x}) = \frac{\omega^2}{c^2}\mathbf{H}(\mathbf{x})$. Show that if you instead eliminate $\mathbf{H}$, you *cannot* get a Hermitian eigenproblem in $\mathbf{E}$ except for the trivial case $\varepsilon =$ constant. Instead, show that you get a *generalized Hermitian eigenproblem*: an equation of the form $\hat{A}\mathbf{E}(\mathbf{x}) = \frac{\omega^2}{c^2}\hat{B}\mathbf{E}(\mathbf{x})$, where *both* $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For *any* generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle \mathbf{E}, \hat{B}\mathbf{E}\rangle > 0$ for all $\mathbf{E}(\mathbf{x}) \neq 0$[1]), show that the eigenvalues (i.e., the solutions of $\hat{A}\mathbf{E} = \lambda\hat{B}\mathbf{E}$) are real and that different eigenfunctions $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $\mathbf{E}$ eigenproblem above was indeed positive definite.

(c) Alternatively, show that $\hat{B}^{-1}\hat{A}$ is Hermitian under a modified inner product $\langle \mathbf{E}, \mathbf{E}\rangle_B = \langle \mathbf{E}, \hat{B}\mathbf{E}\rangle$ for Hermitian $\hat{A}$ and $\hat{B}$ and positive-definite $\hat{B}$; the results from the previous part then follow.

(d) Show that *both* the $\mathbf{E}$ and $\mathbf{H}$ formulations lead to generalized Hermitian eigenproblems with real ω if we allow magnetic materials $\mu(\mathbf{x}) \neq 1$ (but require μ real, positive, and independent of $\mathbf{H}$ or ω).

(e) μ and ε are only ordinary numbers for *isotropic* media. More generally, they are 3×3 matrices (technically, rank 2 tensors)—thus, in an *anisotropic medium,* by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in $\mathbf{E}$ (or $\mathbf{H}$) with real eigenfrequencies ω.

[1] Here, when we say $\mathbf{E}(\mathbf{x}) \neq 0$ we mean it in the sense of generalized functions; loosely, we ignore isolated points where $\mathbf{E}$ is nonzero, as long as such points have zero integral, since such isolated values are not physically observable. See e.g. Gelfand and Shilov, *Generalized Functions*.