

18.369 Midterm Solutions

Spring 2012

Problem 1 :

$$\begin{aligned}
 @) P &= -\frac{i}{2} \operatorname{Re} \int E^* \cdot J = -\frac{i}{2} \operatorname{Re} \int E^* \cdot \left(\frac{-i}{\omega_{M_0}} \hat{A} E \right) \\
 &\quad \text{using } \hat{A} E \\
 &= -\frac{1}{2\omega_{M_0}} \operatorname{Im} \int E^* \cdot (\nabla \times \nabla \times -\frac{\omega^2}{c^2} \epsilon) E \\
 &= \frac{\omega}{2c^2 M_0} \underbrace{\int |E|^2 (\operatorname{Im} \epsilon)}_{> 0} + -\frac{1}{2\omega_{M_0}} \operatorname{Im} \underbrace{\int |\nabla \times E|^2}_{\text{real}} \\
 &\quad > 0 \qquad \qquad \qquad = 0 \\
 &\quad \text{(for } \omega > 0, \operatorname{Im} \epsilon > 0 \text{)}
 \end{aligned}$$

lemma:

$$\begin{aligned}
 &\int E^* \cdot (\nabla \times \nabla \times E) \\
 &= \int (\nabla \times E)^* \cdot (\nabla \times E) \\
 &- \oint \nabla \cdot (E^* \times \nabla \times E) \\
 &= \int |\nabla \times E|^2 \\
 &- \underbrace{\oint E^* \times (\nabla \times E) \cdot dA}_{= 0} \\
 &\text{since } E_{||} = 0 \text{ at boundaries}
 \end{aligned}$$

$$\textcircled{b}) \quad \operatorname{Im} \epsilon > 0 \Rightarrow \operatorname{Im} \frac{1}{\epsilon} < 0 \Rightarrow 2 \operatorname{Re} \left(\frac{i}{\epsilon} \right) > 0$$

$$\frac{i^*}{|\epsilon|^2} = \frac{i}{\epsilon} + \left(\frac{i}{\epsilon} \right)^*$$

$$\Rightarrow \text{consider } \hat{\theta} = i \hat{\theta} :$$

$$\begin{aligned}
 \hat{\theta}^+ + \hat{\theta} &= i \hat{\theta} - i \hat{\theta}^+ \\
 &= \nabla \times \left(\frac{i}{\epsilon} - \frac{i}{\epsilon^*} \right) \nabla \times = 2 \nabla \times \operatorname{Re} \left(\frac{i}{\epsilon} \right) \nabla \times
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \langle H, (\hat{\theta}^+ + \hat{\theta}) H \rangle &= 2 \int \underbrace{|\nabla \times H|^2}_{\geq 0} \operatorname{Re} \left(\frac{i}{\epsilon} \right) \underbrace{\nabla \times H}_{> 0} \quad \text{(integrating by parts in class)} \\
 &\geq 0 \quad \text{Q.E.D. (positive semidef.)}
 \end{aligned}$$

lemma: $\hat{\theta}^* = \nabla \times \frac{1}{\epsilon^*} \nabla \times$

$$\begin{aligned}
 \langle H, \hat{\theta}^* H' \rangle &= \int H^* \cdot \nabla \times \frac{1}{\epsilon^*} \nabla \times H' \\
 &= \int (\nabla \times H)^* \cdot \frac{1}{\epsilon^*} (\nabla \times H') \\
 &- \oint H^* \times \left(\frac{1}{\epsilon^*} \nabla \times H' \right) \cdot dA \xrightarrow{0 \text{ by b.c.}} \\
 &= \int \left(\frac{1}{\epsilon^*} \nabla \times H \right)^* \cdot (\nabla \times H') \\
 &= \int \left(\nabla \times \frac{1}{\epsilon^*} \nabla \times H \right)^* \cdot H' \\
 &- \oint \left(\frac{1}{\epsilon^*} \nabla \times H \right)^* \times H' \cdot dA \xrightarrow{0} \\
 &= \langle \hat{\theta}^* H, H' \rangle \quad \text{(Q.E.D.)}
 \end{aligned}$$

(1b continued)

if the eigenvalues of $\hat{\theta}$ are $\frac{\omega_n^2}{c^2}$, then the eigenvalues of $\hat{O} = i(\hat{\theta})$ are $i\frac{\omega_n^2}{c^2}$, and of $\hat{O} + \hat{O}^+$ are $i\left(\frac{\omega_n^2}{c^2} - \frac{\omega_n^{*2}}{c^2}\right) = -\frac{2}{c^2} \operatorname{Im}(\omega_n^2)$

$\hat{O} + \hat{O}^+$ is positive semidefinite \Rightarrow eigenvalues ≥ 0

$$\Rightarrow \operatorname{Im} \omega_n^2 \leq 0$$

!!



$$2(\operatorname{Re} \omega_n)(\operatorname{Im} \omega_n)$$

\uparrow
assumed
 > 0

$$\Rightarrow \operatorname{Im} \omega_n \leq 0$$

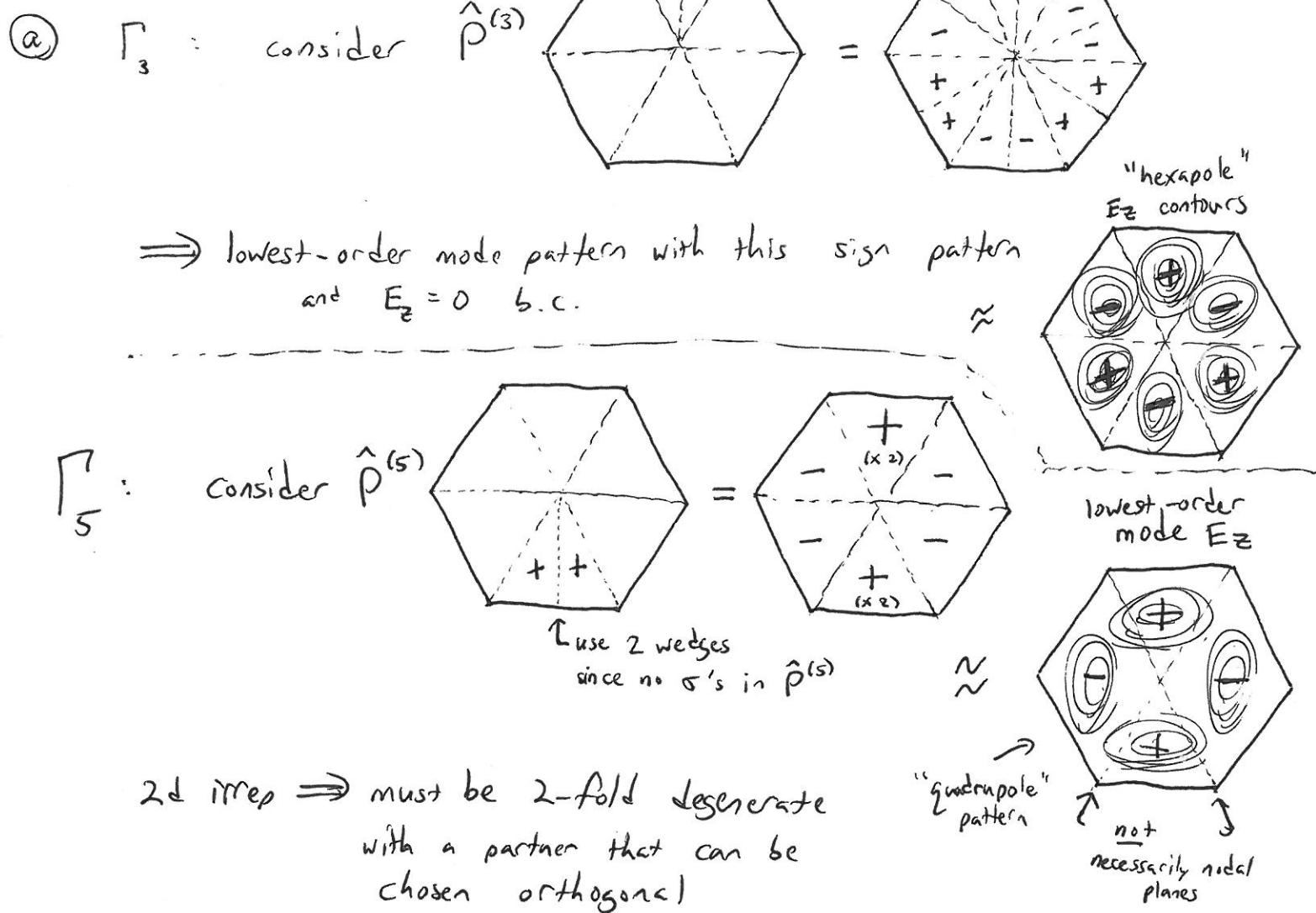
(actually, $= 0$ only for $\omega_n = 0$

since $\langle H, (\hat{O} + \hat{O}^+) H \rangle = 0$
only if $\nabla \times H = 0$)

$$\Rightarrow \text{time dependence } e^{-i\omega_n t} = e^{-i(\operatorname{Re} \omega_n)t + \underbrace{(\operatorname{Im} \omega_n)t}_{< 0}}$$

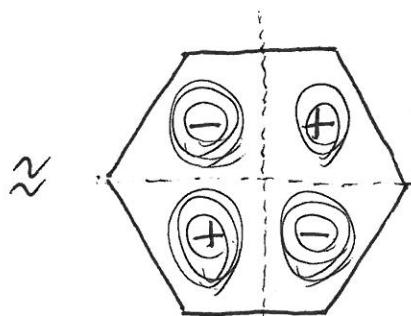
$$= \text{exponentially } \underline{\text{decaying}}$$

Problem 2



quick guess:

similar sign pattern which is \perp by mirror symmetry:
("quadrupole")



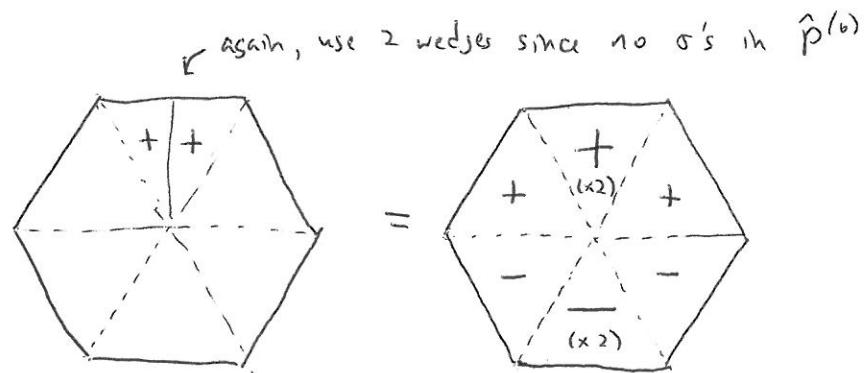
is this Γ_5 ? yes! e.g. we can get this sign pattern from:

$$\hat{P}^{(5)} = \begin{array}{c} +1 \\ +1 \\ +1 \\ +1 \end{array} = \begin{array}{cccccc} -2 & -1 & +1 & +2 & +1 & -1 \\ +1 & +2 & +1 & +1 & -1 & -2 \end{array}$$

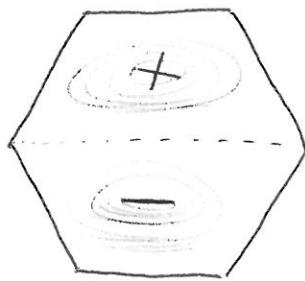
(more technically, the functions xy and x^2-y^2 are partners of Γ_5)

(2a continued)

Γ_6 : consider $\hat{P}^{(6)}$



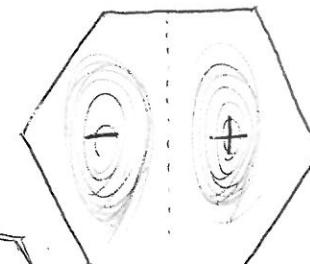
\Rightarrow mode \approx



transform like $\{xy\}$

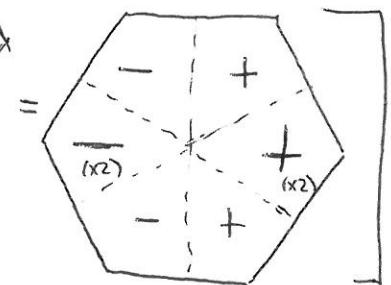
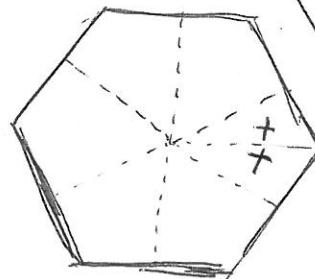
(this irrep =
2d rotation matrices)

\perp by inspection

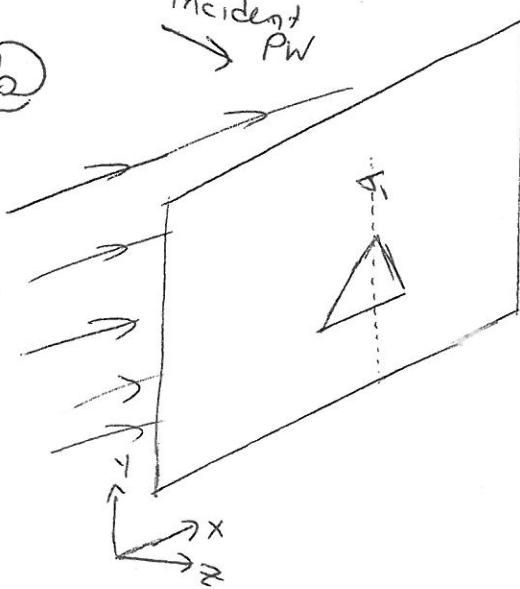


degenerate / orthogonal mode :

e.g. consider $\hat{P}^{(6)}$



b)



simplest answer: incident planewave
is odd with respect to σ_1

\Rightarrow by conservation of NPP,
transmitted waves also must
be odd \Rightarrow x polarized

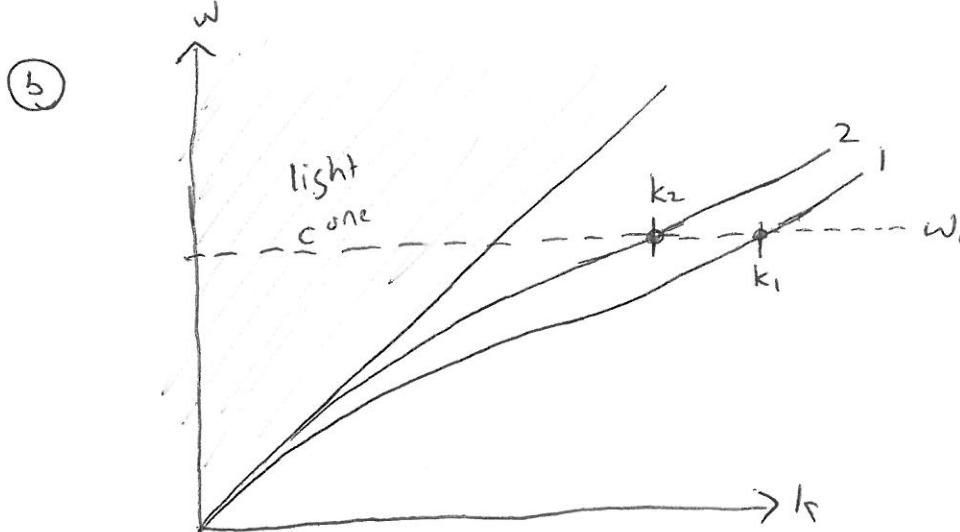
\oplus z-propagating
wave

(any y-polarization
component would be even)

(alternatively, you can use fact that incident wave
is partner of R_3 irrep \Rightarrow transmitted wave must be same ~~(x)~~ partner)

Problem 3

a) Waveguide 1 is lower $\omega \Rightarrow$ more $\epsilon_{ri} \Rightarrow |w_1 > w_2|$



input $\approx \omega_0$ mode at k_1

\Rightarrow output contains $\approx \omega_0$ mode at k_2

but k is conserved up to multiples of $\frac{2\pi}{a}$

$$\Rightarrow k_1 - k_2 = \frac{2\pi}{a} \cdot (\text{integer}) \Rightarrow \boxed{a = \frac{2\pi}{k_1 - k_2} \cdot (\text{integer})}$$

c) similar, but we do not want to couple
 $+k_1$ to $-k_2$ (opposite direction) (or $-k_1$ to $+k_2$)

$$\Rightarrow k_1 + k_2 = \frac{2\pi}{a} \cdot (\text{integer}) \Rightarrow a = \frac{2\pi}{k_1 + k_2} \cdot (\text{integer})$$

$$\Rightarrow \text{smallest such } \boxed{a = \frac{2\pi}{k_1 + k_2}}$$

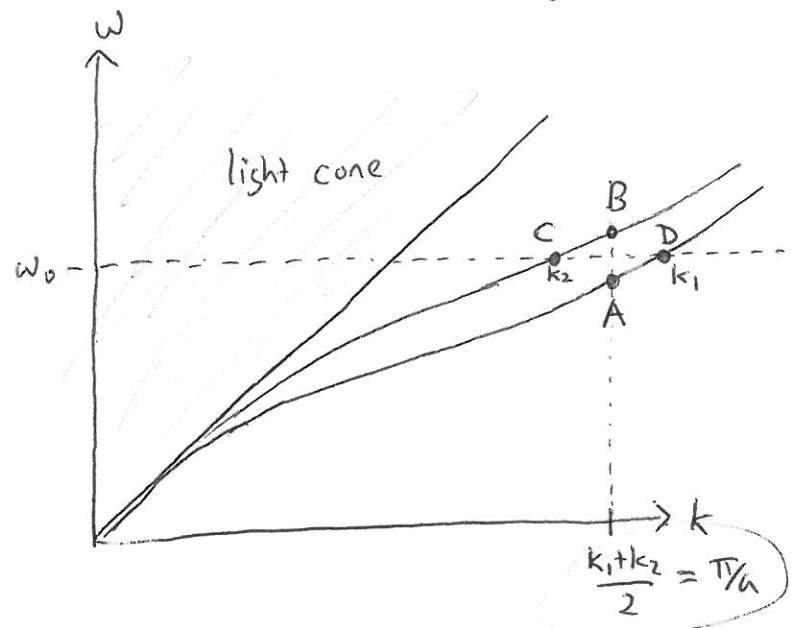
(d)

the choice of part (c) is equivalent to saying that k_1 is "folded" onto k_2

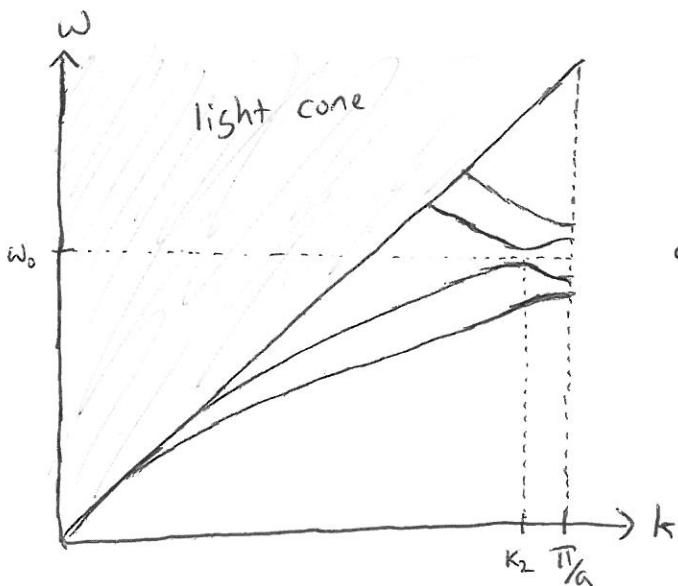
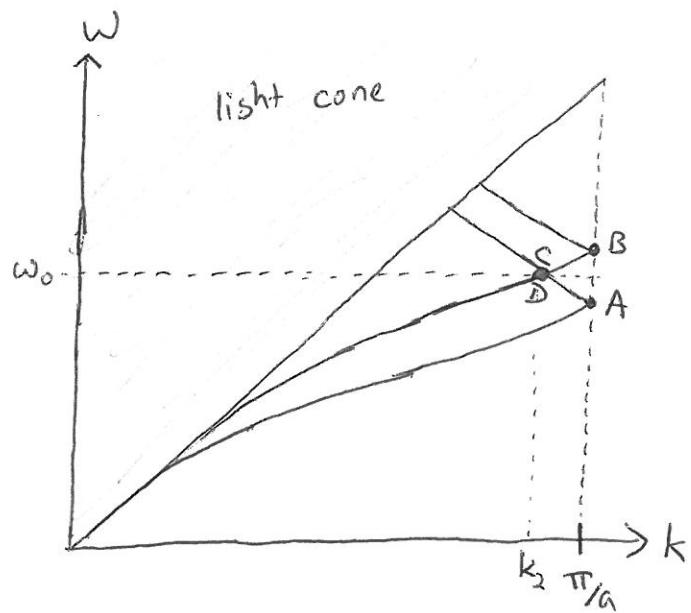
when we add periodicity a & re-label the k axis modulo $\frac{2\pi}{a}$

("folding" at the $0, \frac{\pi}{a}$ Brillouin-zone edges as in class)

" $\frac{k_1+k_2}{2}$!"



artificial folding
(∞ separation,
infinitesimal rods)



in the full MPB calculation of the periodic structure, (small) gaps will open (i.e. anticrossings) at points A , B , and C/D .

ω_0 lies at one such gap
⇒ reflection ... but into waveguide 2
from waveguide 1