18.369 Midterm Exam (Spring 2012)

You have two hours.

Problem 1: Definiteness (30 pts)

Recall from pset 2 that the electric field $\mathbf{E}$ and the current density $\mathbf{J}$ at a frequency ω are related by

$$\hat{A}\mathbf{E} = \left(\nabla\times\nabla\times - \frac{\omega^2}{c^2}\varepsilon\right)\mathbf{E} = i\omega\mu_0\mathbf{J}.$$

Recall also that we "integrate by parts" on $\nabla\times$ using the identity $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})$ along with the divergence theorem $\iiint\nabla\cdot\mathbf{K} = \oiint\mathbf{K}\cdot\mathbf{dA}$.

Take ω to be real and > 0. Suppose you are in a dissipative medium $\varepsilon(\mathbf{x})$ [scalar/isotropic], so that ε is a **complex number** with $\operatorname{Im}\varepsilon > 0$ everywhere (for $\omega > 0$), enclosed within a perfect-metal box of some arbitrary smooth shape (so that the surface-parallel component of $\mathbf{E}$ is zero at the metal walls).

(a) Show that the time-average power P expended by any current $\mathbf{J}$, given by $P = -\frac{1}{2}\operatorname{Re}\int\mathbf{E}^*\cdot\mathbf{J}$ from class (via Poynting's theorem), is necessarily positive ($P > 0$).

(b) Relate $\Theta = \nabla\times\frac{1}{\varepsilon}\nabla\times$ to some operator $\hat{O}$ such that $\hat{O}+\hat{O}^\dagger$ is positive-definite under the usual inner product $\langle\mathbf{H},\mathbf{H}'\rangle = \int\mathbf{H}^*\cdot\mathbf{H}'$ (where the boundary conditions are that the surface-parallel component of $\nabla\times\mathbf{H}$ vanishes at the walls). From class, the eigenvalues of Θ are $\frac{\omega_n^2}{c^2}$, where ω_n are the eigenfrequencies, here taken to have positive real parts. From the definiteness of $\hat{O}+\hat{O}^\dagger$, show that the eigensolutions are decaying in time (as you would expect in a dissipative system).

Problem 2: Symmetry (30 points)

The symmetry operations for the regular hexagon (symmetry group C_{6v}) and the equilateral triangle (symmetry group C_{3v}, from problem set 2) are shown in figure 1. The corresponding character tables are:

C_{6v}	E	$2C_6$	$2C_3$	C_2	3σ	$3\sigma'$
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1
Γ_3	1	-1	1	-1	1	-1
Γ_4	1	-1	1	-1	-1	1
Γ_5	2	-1	-1	2	0	0
Γ_6	2	1	-1	-2	0	0

C_{3v}	E	$2C_3$	3σ
R_1	1	1	1
R_2	1	1	-1
R_3	2	-1	0

(where we have called the C_{6v} representations $\Gamma_{1,\ldots,6}$ and the C_{3v} representations $R_{1,\ldots,3}$ to distinguish them).

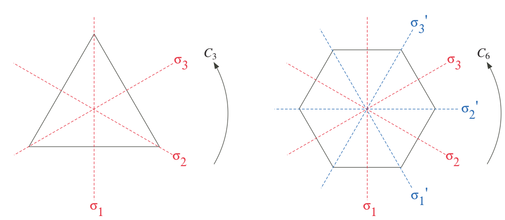

Figure 1: Left: symmetry operations (C_{3v}) for the triangle (three mirror planes and 3-fold rotations). Right: symmetry operations (C_{3v}) for the hexagon (six mirror planes and 6-fold rotations).

(a) Suppose you have a 2d (xy) perfect metallic cavity, filled with air, in the shape of a regular hexagon, and we consider the TM polarization (E_z only, $E_z = 0$ at walls). Sketch contour plots (label $\pm$) of the lowest-frequency eigenmodes corresponding to Γ_3, Γ_5, and Γ_6. (Show degenerate modes as orthogonal pairs.) [Hint: to get a feel for things, try projecting a function that is positive in one or two of the triangular wedges and zero elsewhere.]

(b) Suppose that we have an infinite metal plate (lying in the xy plane with some finite thickness in z) suspended in air, with a hole punched through it. The hole is in the shape of an equilateral triangle, with one side of the triangle falling on the x axis. We now shine a normal-incident planewave (propagating in the $+z$ direction) at the plate, polarized with its electric field in the $\pm x$ direction. If we decompose (Fourier transform) the transmitted wave (propagating in the air on the other side of the plate) into planewaves, some portion will be a planewave propagating in the $+z$ direction. Explain why this forward-transmitted wave must be polarized in the $\pm x$ direction.

Problem 3: Bands (30 points)

Suppose that we have two dielectric waveguides 1 and 2 in air, in two dimensions for the TM polarization, with different widths so that their band diagrams are slightly different: the dispersion relations $\omega(k)$ of the fundamental modes of both waveguides are shown in figure 2.

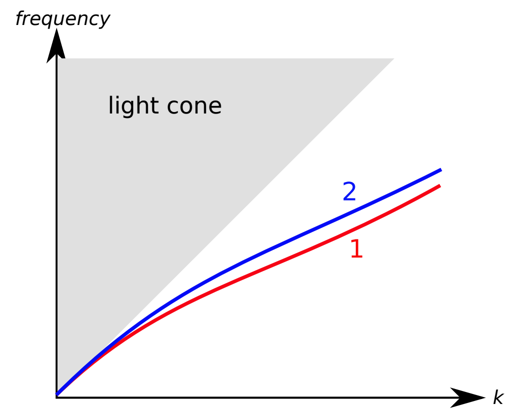

Figure 2: Dispersion relations $\omega(k)$ of fundamental modes of two different waveguides 1 and 2, both in air (hence, same light cone $\omega \geq c|k|$).

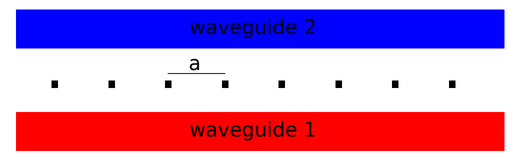

Figure 3: Two different waveguides 1 and 2 alongside one another, between which we put a sequence of small dielectric posts with period a. Assume that the waveguides are far enough apart, and the posts are small enough, that the waveguide eigenfunctions are only weakly perturbed.

(a) Suppose that both waveguides are made of the same material ε_{hi} but have different widths w_1 and w_2. From the band diagram, which of w_1 and w_2 is bigger?

(b) Suppose that we put the two waveguides alongside one another (but far enough apart that their modes only slightly overlap). In between them, we put a sequence of small dielectric posts with period a, as shown in figure 3. Assume that the waveguides are far enough apart, and the posts are small enough, that the waveguide eigenfunctions are only *weakly perturbed* from those of the isolated waveguides. If we start with the fundamental mode propagating in waveguide 1 at a frequency ω_0, and after propagating for some distance we observe that much of the energy is now propagating in waveguide 2 (in the *same* direction), what can you say about the period a? (Relate a to some $k's$ that you label schematically in the band diagram.)

(c) Suppose that you have the same situation as in the previous part, but now you want to start in waveguide 1 at ω_0 and end up in waveguide 2 propagating in the *opposite* direction. What would you choose a to be (again in terms of labelled k's)? (If there is more than one possible a, give the smallest choice.)

(d) If you calculated the band diagram for this whole structure (i.e. both waveguides + posts) together in MPB, with the a from part (c), sketch what the band diagram should look like for $k \in [0, \pi/a]$ (label ω_0, and label corresponding points in figure 2 and your new band diagram with A, B, C, ... as needed to make the quantitative relationships clear). It might be clearer to do this in two steps: first plot the "folded" band diagram for a period a but assuming the waveguides are infinitely far apart and the posts are infinitesimal (i.e. no perturbation to the solutions, just a relabeling), and then plot what happens for a finite separation with finite-size posts.