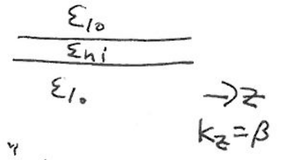
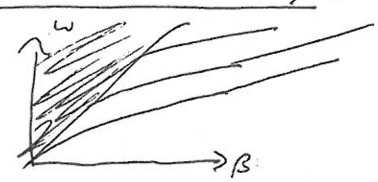
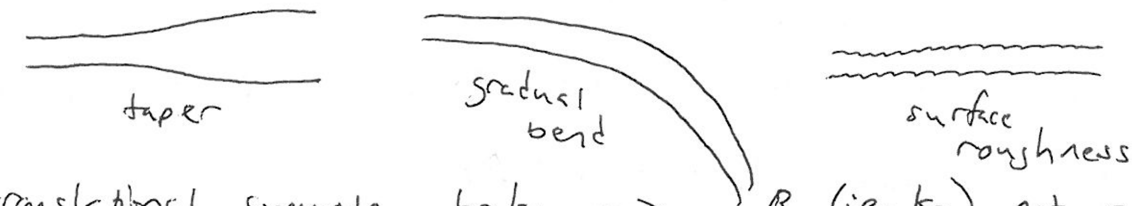
Waveguide Losses – Mode Coupling
(+ coupled-mode theory)

* 2d waveguide examples





various "small perturbations":



translational symmetry broken $\Rightarrow$ β (i.e. k_z) not conserved

$\Rightarrow$ even if we in input 1 mode (β, ω)
we may get out multiple modes (β', ω)
[including reflected mode $(-\beta, \omega)$]

↑ same frequency since still linear time-invariant

* we want to analyze this, exploiting fact that perturbation is weak — waveguide is nearly uniform

$\Rightarrow$ expand field $|\psi\rangle$ in modes $|n\rangle$ of uniform waveguide:

$$|\psi\rangle = \sum_n \underbrace{c_n(z)}_{\text{expansion coef's dep. on } z} \; \underbrace{|n\rangle}_{\text{what are these?}}$$

+ solve for $c_n(z)$ approximately . . .

what are these? they must be eigenmodes at different β, but same ω

$\Rightarrow$ not $\hat{\Theta}_k |H\rangle = \frac{\omega(k)^2}{c^2} |H\rangle$ eigenproblem!

* Maxwell at fixed ω:

$$\nabla \times \vec{E} = i\omega\mu\vec{H}/c$$
$$\nabla \times \vec{H} = -i\frac{\omega}{c}\varepsilon\vec{E}$$

automatic at $\omega \neq 0$
$$\nabla \cdot \mu H = 0 \;;$$
$$\nabla \cdot \varepsilon E = 0$$

write $\nabla = \underbrace{\nabla_t}_{\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}} + \frac{\partial}{\partial z}\hat{z}$

where t = "transverse" = xy components

$$\vec{E} = \vec{E}_t + E_z\hat{z}$$
$$\vec{H} = \vec{H}_t + H_z\hat{z}$$

recall: $\nabla\times E = \begin{vmatrix} \hat x & \hat y & \hat z \\ \underbrace{\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}}_{t} & & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \left.\nabla_t\times\vec E_t\right\}z$

$$\left.\begin{aligned} &+\frac{\partial}{\partial z}\hat z\times\vec E_t \\ &+\nabla_t\times(\hat z E_z)\end{aligned}\right\}xy$$

$$\Rightarrow\ i\frac{\omega}{c}\mu H_z\hat z = \nabla_t\times E_t \quad , \Rightarrow\ \nabla_t\times\hat z H_z = -\frac{ic}{\omega}\nabla_t\times\frac{1}{\mu}\nabla_t\times E_t$$

$$i\frac{\omega}{c}\mu\vec H_t = \underbrace{\qquad\qquad}_{\nabla_t\times\hat z E_z} + \frac{\partial}{\partial z}\hat z\times\vec E_t$$

$$-i\frac{\omega}{c}\varepsilon E_z\hat z = \nabla_t\times H_t \Rightarrow \nabla_t\times\hat z E_z = \frac{ic}{\omega}\nabla_t\times\frac{1}{\varepsilon}\nabla_t\times H_t$$

$$-i\frac{\omega}{c}\varepsilon\vec E_t = \underbrace{\qquad\qquad}_{\nabla_t\times\hat z H_z} + \frac{\partial}{\partial z}\hat z\times\vec H_t$$

$$\Rightarrow\ \left(-i\frac{\omega}{c}\varepsilon + \frac{ic}{\omega}\nabla_t\times\frac{1}{\mu}\nabla_t\times\right)E_t = \frac{\partial}{\partial z}\hat z\times\vec H_t$$

$$\left(i\frac{\omega}{c}\mu - \frac{ic}{\omega}\nabla_t\times\frac{1}{\varepsilon}\nabla_t\times\right)H_t = \frac{\partial}{\partial z}\hat z\times\vec E_t$$

$$\Rightarrow\ \underbrace{\begin{pmatrix}\frac{\omega\varepsilon}{c}-\frac{c}{\omega}\nabla_t\times\frac{1}{\mu}\nabla_t\times & 0 \\ 0 & \frac{\omega\mu}{c}-\frac{c}{\omega}\nabla_t\times\frac{1}{\varepsilon}\nabla_t\times\end{pmatrix}}_{=\hat A}\underbrace{\begin{pmatrix}\vec E_t \\ \vec H_t\end{pmatrix}}_{=|\psi\rangle} = -i\frac{\partial}{\partial z}\underbrace{\begin{pmatrix} & -\hat z\times \\ \hat z\times & \end{pmatrix}}_{=\hat B}\begin{pmatrix}\vec E_t \\ \vec H_t\end{pmatrix}$$

$$\Rightarrow\ \hat A\,|\psi\rangle = -i\frac{\partial}{\partial z}\hat B\,|\psi\rangle$$

compare: $\hat H|\psi\rangle = +i\frac{\partial}{\partial t}|\psi\rangle$
Schrödinger

define inner product $\langle\psi|\psi'\rangle = \iint\left(\vec E_t^*\cdot\vec E_t' + \vec H_t^*\cdot\vec H_t'\right)dx\,dy$
at a fixed z

$\Rightarrow\ \hat A,\ \hat B$ are Hermitian for real ε, μ

— in fact, they are real-symmetric operators
— they are not positive definite (eigenvalues of $\hat B$ are ± 1)