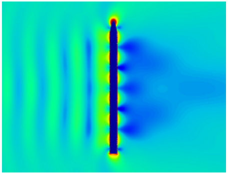
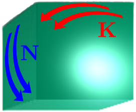


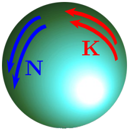
Surface Integral Equations and the Boundary Element Method

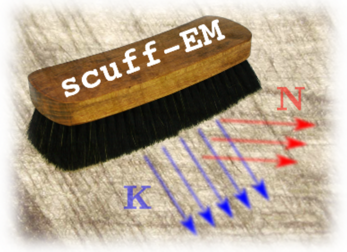
Homer Reid

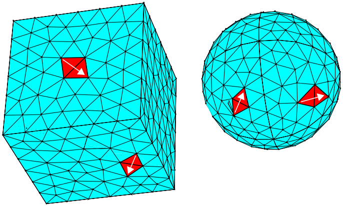
18.369 Guest Lecture

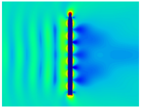
3/23/2012





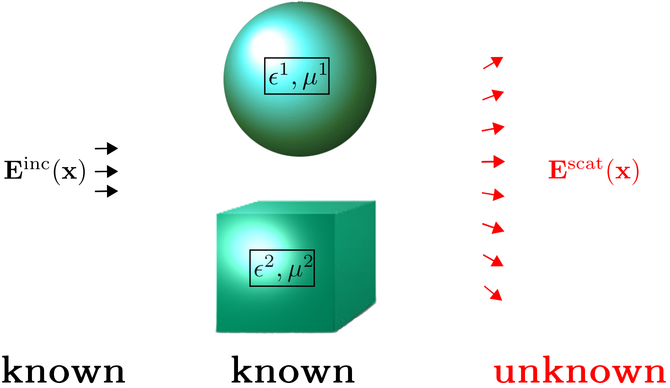


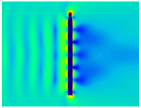




Electromagnetic Scattering Problems

We have some known **incident field** (such as a plane wave), scattering from some known **geometry** (including objects of known shapes and materials) and we want to know the scattered fields. (Note: all quantities $\sim e^{-i\omega t}$.)

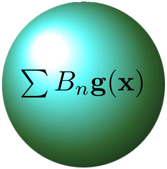




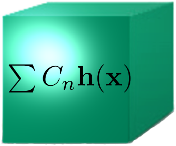
Methods for Solving EM Scattering Problems, 1

Expansions in special functions

Write the fields inside and outside the scatterer as expansions in sets of known Maxwell solutions (in some convenient coordinate system) and match coefficients.



$$\mathbf{E}(\mathbf{x}) = \sum A_n \mathbf{f}(\mathbf{x})$$



Advantages:

- Exploits known Maxwell solutions $\Longrightarrow$ **efficient**

Disadvantages:

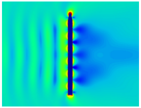
- Only works for a small number of geometries $\Longrightarrow$ **not general**.

One Sphere: "Mie scattering"

$$\mathbf{f}(\mathbf{x}) \sim j_l(r) Y_{lm}(\theta, \phi)$$

Planar Slab: "Fresnel Coefficients"

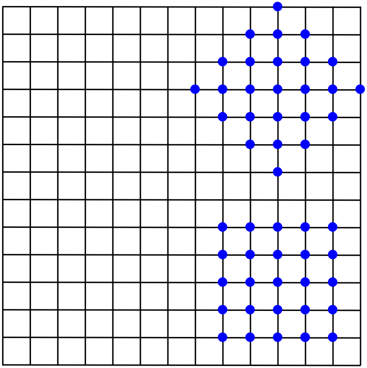
$$\mathbf{f}(\mathbf{x}) \sim e^{i\mathbf{k}\cdot\mathbf{x}}$$



Methods for Solving EM Scattering Problems, 2

Finite-Difference Method

- Discretize the geometry onto a grid (each grid point can have different ϵ, μ)
- Write Maxwell's equations using finite-difference approximations to derivatives
- Solve sparse linear system for the $\mathbf{E}$-field values at grid points



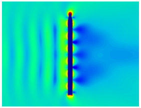
$$\Big[\nabla\times\nabla\times - k^2\Big]\mathbf{E} = -i\omega\mathbf{J} \Longrightarrow \begin{pmatrix} \mathbf{M} \end{pmatrix}\begin{pmatrix} \mathbf{E}_1 \\ \vdots \\ \mathbf{E}_n \end{pmatrix} = i\omega \begin{pmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_n \end{pmatrix}$$

Advantages:

- Allows different ϵ, μ at each grid point $\longrightarrow$ **general**
- Relatively **easy to implement**

Disadvantages:

- Does not make use of known Maxwell solutions $\longrightarrow$ **not the most efficient method**
- If we need to evaluate the scattered fields far from the scattering objects, we have to discretize the entire space between the objects and the evaluation point. $\longrightarrow$ **Seems wasteful.**



Methods for Solving EM Scattering Problems, 3

Surface-Integral-Equation (SIE) Method

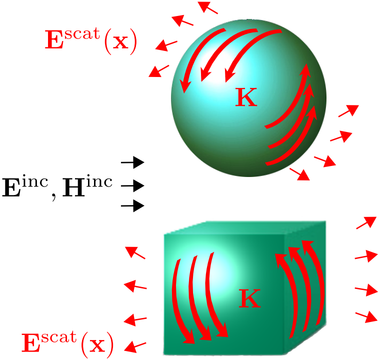
- First compute the surface current distribution $\mathbf{K}(\mathbf{x})$ induced by the incident field
- Then compute the scattered fields using $\mathbf{K}(\mathbf{x})$ and known Maxwell solutions:

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \oint_{\mathcal{S}} \mathbf{G}(\mathbf{x}-\mathbf{x}')\mathbf{K}(\mathbf{x}')d\mathbf{x}'$$

where $\mathbf{G}$ is the solution to

$$\Big[\nabla \times \nabla \times - k^2\Big]\mathbf{G}(\mathbf{r}) = -i\omega\mathbf{1}\delta(\mathbf{r});$$

$\mathbf{G}$ (the "dyadic Green's function") is **known in closed form**

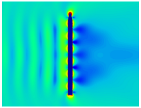


Advantages:

- Exploits known Maxwell solutions $\Longrightarrow$ **efficient**
- Allows scatterers of arbitrary shapes and arbitrary (homogeneous) materials $\Longrightarrow$ **general**
- Unknown quantities confined to object surfaces, not everywhere in space $\Longrightarrow$ **not wasteful**

Disadvantages:

- Difficult to implement
- Restricted to homogeneous scatterers, i.e. piecewise-constant ϵ, μ



SIE Formulation of Scattering Problems

Consider a perfectly electrically conducting (PEC) scatterer in vacuum.

The incident field induces a surface electric current density $\mathbf{K}(\mathbf{x})$ on the object surface.

Surface current density $\mathbf{K}$: units of $\frac{\text{current}}{\text{length}}$

$$\underbrace{\mathbf{J}(\mathbf{x}_\parallel, z)}_{\text{volume current}} = \underbrace{\mathbf{K}(\mathbf{x}_\parallel)}_{\text{surface current}} \cdot \delta(z)$$

Once we know $\mathbf{K}(\mathbf{x})$, we can compute the scattered $\mathbf{E}$−field anywhere we like:

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \oint_{\mathcal{S}} \mathbf{G}(\mathbf{x}-\mathbf{x}')\mathbf{K}(\mathbf{x}')d\mathbf{x}'$$

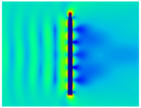
$$G_{ij}(\mathbf{r}) = \frac{e^{ikr}}{4\pi k^2 r^3}\left\{\Big[1-ikr+(ikr)^3\Big]\delta_{ij} + \Big[-3+3ikr-(ikr)^2\Big]\frac{\mathbf{r}_i\mathbf{r}_j}{r^2}\right\} \qquad \left(r=|\mathbf{r}|, \quad k=\frac{\omega}{c}\right)$$

We determine $\mathbf{K}(\mathbf{x})$ by requiring that the total tangential $\mathbf{E}$-field vanish at the object surface:

$$\Big[\mathbf{E}^{\text{inc}}(\mathbf{x}) + \mathbf{E}^{\text{scat}}(\mathbf{x})\Big]_\parallel = 0 \qquad \Longrightarrow \qquad \boxed{\oint_{\mathcal{S}} \mathbf{G}_\parallel(\mathbf{x},\mathbf{x}')\mathbf{K}(\mathbf{x}')d\mathbf{x}' = -\mathbf{E}^{\text{inc}}_\parallel(\mathbf{x})}$$

(for points $\mathbf{x}$ on object surfaces) — "electric field integral equation" (EFIE)

The EFIE is an integral equation for $\mathbf{K}(\mathbf{x})$ in terms of $\mathbf{E}^{\text{inc}}$.



Numerical Solution of SIEs

The boundary element method (BEM)

Given $\mathbf{E}^{\text{inc}}(\mathbf{x})$, want to find $\mathbf{K}(\mathbf{x})$ that solves the EFIE: $\oint_{\mathcal{S}} \mathbf{G}_{\parallel}(\mathbf{x},\mathbf{x}')\mathbf{K}(\mathbf{x}')d\mathbf{x}' = -\mathbf{E}_{\parallel}^{\text{inc}}(\mathbf{x})$

Idea: (1) **expand** $\mathbf{K}(\mathbf{x})$ in some convenient set of N basis functions $\Longrightarrow$ N unknown coefficients

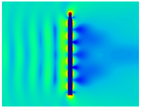
$$\mathbf{K}(\mathbf{x}) = \sum_{n=1}^{N} k_n \mathbf{f}_n(\mathbf{x}), \qquad \Big\{\mathbf{f}_n(\mathbf{x})\Big\} = \left(\begin{array}{c}\text{tangential vector-valued basis functions}\\ \text{defined on the object surface}\end{array}\right)$$

Idea: (2) **test** (inner-product) the EFIE with each basis function $\Longrightarrow$ N equations

$$\left\langle \mathbf{f}_m, \oint \mathbf{G}\cdot \underbrace{\mathbf{K}}_{\sum k_n \mathbf{f}_n} dA \right\rangle = -\left\langle \mathbf{f}_m, \mathbf{E}^{\text{inc}} \right\rangle \qquad \Longrightarrow \qquad \Big(\ \mathbf{M}\ \Big)\begin{pmatrix} k_1 \\ \vdots \\ k_N \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

$N \times N$ linear system ("BEM system")

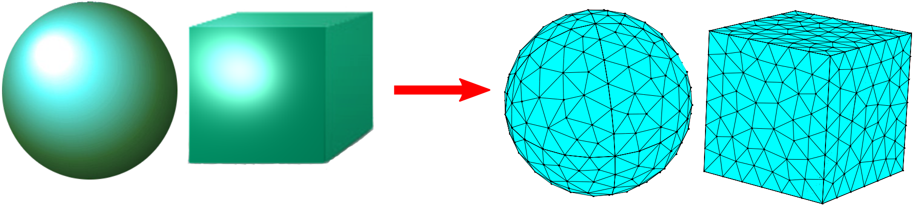
Matrix elements: $M_{mn} = \Big\langle \mathbf{f}_m \Big| \mathbf{G} \Big| \mathbf{f}_n \Big\rangle$ RHS vector: $v_m = -\Big\langle \mathbf{f}_m \Big| \mathbf{E}^{\text{inc}} \Big\rangle$



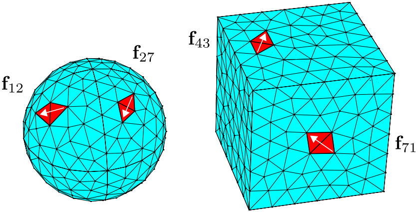
Basis functions for SIE/BEM solvers

One choice for compact 3D objects: “RWG basis functions”

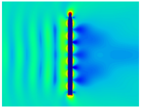
Begin by discretizing (“meshing”) object surfaces into triangles:



Associate one basis function with each internal edge:



- These are “RWG basis functions” (named for their inventors: Rao, Wilton, Glisson)
- # of basis functions $N \propto$ # of triangles
- As we refine the discretization (shrink the triangles), the discretization errors decrease, but the cost of solving the linear system grows like N^3



Steps in a BEM Scattering Calculation

For a compact 3D scattering problem using RWG basis functions

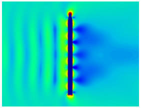
1. Discretize object surfaces into triangles.
   - A well-studied problem; high-quality free software packages are available.
2. Analyze the surface mesh and assign one basis function $\mathbf{f}_n(\mathbf{x})$ to each interior edge.
   - Some minor computational work; not too challenging.
3. Most difficult step: Assemble the BEM matrix $\mathbf{M}$ and RHS vector $\mathbf{v}$.

$$M_{mn} = \Big\langle \mathbf{f}_m \Big| \mathbf{G} \Big| \mathbf{f}_n \Big\rangle, \qquad v_m = -\Big\langle \mathbf{f}_m \Big| \mathbf{E}^{\text{inc}} \Big\rangle$$

4. Solve the linear system $\mathbf{Mk} = \mathbf{v}$ for the surface-current expansion coefficients $\{k_n\}$.
   - For $N \lesssim 10,000$, use standard linear algebra software (LAPACK).
5. Use the surface current density $\mathbf{K}(\mathbf{x}) = \sum k_n \mathbf{f}_n(\mathbf{x})$ to compute the scattered fields.

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \sum_n k_n \int \mathbf{G}^{\text{EE}}(\mathbf{x},\mathbf{x}')\mathbf{f}_n(\mathbf{x}')d\mathbf{x}', \qquad \mathbf{H}^{\text{scat}}(\mathbf{x}) = \sum_n k_n \int \mathbf{G}^{\text{ME}}(\mathbf{x},\mathbf{x}')\mathbf{f}_n(\mathbf{x}')d\mathbf{x}',$$

where $\mathbf{G}^{\text{EE}}$ is what we called "$\mathbf{G}$" before and $\mathbf{G}^{\text{ME}} \sim \nabla \times \mathbf{G}^{\text{EE}}$.



Why is it so hard to assemble the BEM matrix?

Consider a scattering geometry with surfaces discretized into $N \sim 10,000$ triangles.

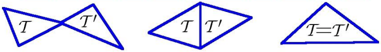
1. We have N^2=100 million matrix elements.
2. Each matrix element involves a 4 dimensional integral (surface integrals over two triangles) that must be evaluated numerically.
3. A sizeable fraction of these are singular integrals.

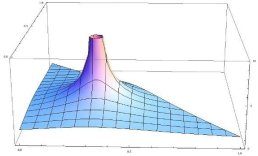
$$\boxed{M_{mn} = \Big\langle \mathbf{f}_m \Big| \mathbf{G} \Big| \mathbf{f}_n \Big\rangle}$$

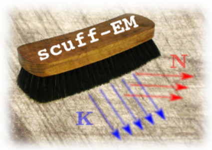
$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1N} \\ M_{21} & M_{22} & M_{23} & \cdots & M_{2N} \\ M_{31} & M_{32} & M_{33} & \cdots & M_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & M_{N3} & \cdots & M_{NN} \end{pmatrix}$$

10,000 (rows) × 10,000 (columns)

$$\int_{\mathcal{T}} d\mathbf{x} \int_{\mathcal{T}'} d\mathbf{x}'\, h(\mathbf{x}, \mathbf{x}')g\big(|\mathbf{x} - \mathbf{x}'|\big)$$







SCUFF-EM: Free, open-source software for boundary-element analysis of problems in computational physics and engineering

SCUFF-EM is a free, open-source software package for analysis of electromagnetic scattering problems using the boundary-element method (BEM). (The BEM is also known as the "method of moments.")

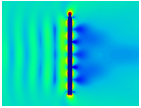
The SCUFF-EM suite consists of two components: a *core library* that implements the essential algorithms of the boundary-element method, and a set of *application programs* built atop the core library for solving specific problems in various fields of physics and engineering.

The core library, LIBSCUFF, is written in C++ but may also be accessed from PYTHON and MATLAB. Extensive documentation of the programming interfaces is available from the links below.

The application programs, implemented as console-based command-line utilities, include tools for **(1)** *general electromagnetic scattering* of arbitrary incident fields from compact or periodically extended scatterers; **(2)** computation of *Casimir forces* and *Casimir-Polder potentials* in complex geometric and material configurations; and **(3)** modeling of *RF and microwave devices*, including computation of multiport network parameters and radiated fields for antennas, lumped elements, and other RF devices.

The entire SCUFF-EM suite is free software distributed under the GNU GPL.

SCUFF-EM stands for ***S**urface **CU**rrent/**F**ield **F**ormulation of **E**lectro**M**agnetism*. This is a reference to the underlying solution methodology used by SCUFF-EM and other BEM solvers, in which we solve



SIE/BEM Techniques for Non-PEC Geometries

For non-PEC geometries we must introduce effective magnetic surface currents

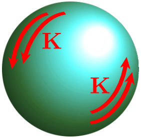
For PEC scatterers, the SIE/BEM procedure reflects a physical reality: the currents induced by the incident field are confined to the object surface.

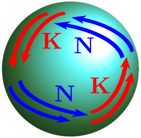
For general (non-PEC) scatterers, this is no longer true: the incident field induces currents throughout the volume of the scatterer.

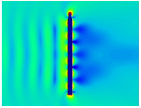
Two options:

1. Volume integral equation: Write an integral equation for the volume electric current distribution $\mathbf{J}(\mathbf{x})$ throughout the bulk of the scatterer.
2. Surface integral equation: Write an integral equation for effective electric **and magnetic** surface currents $\mathbf{K}(\mathbf{x}), \mathbf{N}(\mathbf{x})$ on the surface of the scatterer.

	PEC	Non-PEC
Physics	Surface electric current $\mathbf{K}$	Volume electric current $\mathbf{J}$
Mathematics	Surface electric current $\mathbf{K}$	Surface electric and magnetic currents $\mathbf{K}, \mathbf{N}$





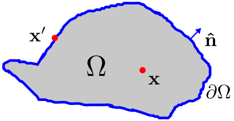


Effective Surface Currents for non-PEC Geometries

The Stratton-Chu equations

Recall Green's theorem: For a **scalar field** ϕ satisfying **Laplace**, knowledge of ϕ (or $\frac{\partial \phi}{\partial \hat{\mathbf{n}}}$) on the boundary $\partial\Omega$ of a closed source-free region Ω suffices to recover ϕ everywhere in the interior.

$$\phi(\mathbf{x}) = \oint_{\partial\Omega} G(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}')dA$$



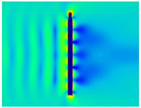
The Stratton-Chu equations generalize Green's theorem to the case of **vector fields** satifying **Maxwell**: knowledge of tangential $\mathbf{E}$, $\mathbf{H}$ on $\partial\Omega$ suffices to recover $\mathbf{E}$ and $\mathbf{H}$ throughout Ω.

$$\mathbf{E}(\mathbf{x}) = \oint_{\partial\Omega} \Big\{ \mathbf{G}^{\mathsf{EE}}(\mathbf{x}, \mathbf{x}')\Big[\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{x}')\Big] + \mathbf{G}^{\mathsf{EM}}(\mathbf{x}, \mathbf{x}')\Big[-\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{x}')\Big] \Big\} dA$$

$$\mathbf{H}(\mathbf{x}) = \oint_{\partial\Omega} \Big\{ \mathbf{G}^{\mathsf{ME}}(\mathbf{x}, \mathbf{x}')\Big[\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{x}')\Big] + \mathbf{G}^{\mathsf{MM}}(\mathbf{x}, \mathbf{x}')\Big[-\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{x}')\Big] \Big\} dA$$

The source quantities that enter the Stratton-Chu equations are $\hat{\mathbf{n}} \times \mathbf{H}$ and $-\hat{\mathbf{n}} \times \mathbf{E}$. Think of these as effective surface currents:

$$\boxed{\mathbf{K}^{\mathsf{eff}}(\mathbf{x}) \equiv \hat{\mathbf{n}} \times \mathbf{H}, \qquad \mathbf{N}^{\mathsf{eff}}(\mathbf{x}) \equiv -\hat{\mathbf{n}} \times \mathbf{E}.}$$



BEM Formulation for non-PEC Scatterers

Generalizing the EFIE

Fields inside and outside the scatterer:

$$\left[\begin{array}{c}\mathbf{E}^{\text{in}}(\mathbf{x})\\ \mathbf{H}^{\text{in}}(\mathbf{x})\end{array}\right] = -\oint_{\partial\Omega}\Big[\mathbf{G}^{\text{in}}(\mathbf{x},\mathbf{x}')\Big]\left[\begin{array}{c}\mathbf{K}(\mathbf{x}')\\ \mathbf{N}(\mathbf{x}')\end{array}\right]d\mathbf{x}'$$

$$\left[\begin{array}{c}\mathbf{E}^{\text{out}}(\mathbf{x})\\ \mathbf{H}^{\text{out}}(\mathbf{x})\end{array}\right] = +\oint_{\partial\Omega}\Big[\mathbf{G}^{\text{out}}(\mathbf{x},\mathbf{x}')\Big]\left[\begin{array}{c}\mathbf{K}(\mathbf{x}')\\ \mathbf{N}(\mathbf{x}')\end{array}\right]d\mathbf{x}' + \left[\begin{array}{c}\mathbf{E}^{\text{inc}}(\mathbf{x})\\ \mathbf{H}^{\text{inc}}(\mathbf{x})\end{array}\right]$$

Match tangential fields at the scatterer surface (for points $\mathbf{x}\in\partial\Omega$):

$$\mathbf{E}^{\text{in}}_{\parallel}(\mathbf{x}) = \mathbf{E}^{\text{out}}_{\parallel}(\mathbf{x})$$

$$\mathbf{H}^{\text{in}}_{\parallel}(\mathbf{x}) = \mathbf{H}^{\text{out}}_{\parallel}(\mathbf{x})$$

$$\Longrightarrow \boxed{\oint_{\partial\Omega}\Big[\mathbf{G}^{\text{out}}+\mathbf{G}^{\text{in}}\Big]_{\parallel}\left[\begin{array}{c}\mathbf{K}(\mathbf{x}')\\ \mathbf{N}(\mathbf{x}')\end{array}\right]d\mathbf{x}' = -\left[\begin{array}{c}\mathbf{E}^{\text{inc}}(\mathbf{x})\\ \mathbf{H}^{\text{inc}}(\mathbf{x})\end{array}\right]_{\parallel}}$$

Integral equation for $\mathbf{K},\mathbf{N}$ in terms of $\mathbf{E}^{\text{inc}},\mathbf{H}^{\text{inc}}$

Discretize by expanding $K(\mathbf{x})=\sum k_n\mathbf{f}_n(\mathbf{x})$, $\quad N(\mathbf{x})=\sum n_n\mathbf{f}_n(\mathbf{x})$:

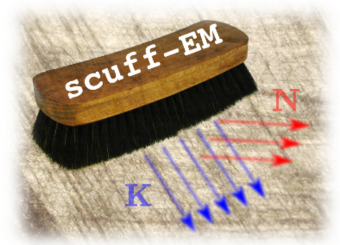
$$\boxed{\Big(\ \mathbf{M}\ \Big)\left(\begin{array}{c}k_n\\ n_n\end{array}\right)=\left(\begin{array}{c}v_n^{\text{E}}\\ v_n^{\text{H}}\end{array}\right)}$$

("PMCHW Formulation")

$\Longrightarrow 2N\times 2N$ linear system for the expansion coefficients $\{k_n, n_n\}$

SCUFF-EM: An open-source BEM code suite

Surface-**C**urrent / **F**ield **F**ormulation of **E**lectro**M**agnetism



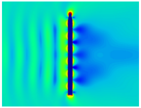
http://homerreid.com/scuff-EM

Features currently available:

- Scattering from **compact 3D objects** of arbitrary shapes
- **Arbitrary** user-specified frequency-dependent ϵ, μ (isotropic, linear, piecewise constant)
- Linux/Athena **command-line interface** to scattering code
- **C++ interface** to scattering code
- Application modules: Casimir forces, RF device modeling

Features coming soon:

- **Python / Matlab** interfaces to scattering codes
- Scattering from **periodic geometries**



Solving scattering problems with SCUFF-EM

Scattering of a gaussian laser beam from a silver nanotip

```
scuff-scatter --geometry Tip.scuffgeo
              --Omega 2.3
              --pwDirection 0 0 1
              --pwPolarization 1 0 0
              --EPFile MyEvalPoints
```

Tip mesh: