

## 18.369 Problem Set 1

Due Friday, 12 February 2010.

### Problem 1: Adjoints and operators

- (a) We defined the adjoint  $\dagger$  of operators  $\hat{O}$  by:  $\langle H_1, \hat{O}H_2 \rangle = \langle \hat{O}^\dagger H_1, H_2 \rangle$  for all  $H_1$  and  $H_2$  in the vector space. Show that for a *finite-dimensional* Hilbert space, where  $H$  is a column vector  $h_n$  ( $n = 1, \dots, d$ ),  $\hat{O}$  is a square  $d \times d$  matrix, and  $\langle H^{(1)}, H^{(2)} \rangle$  is the ordinary conjugated dot product  $\sum_n h_n^{(1)*} h_n^{(2)}$ , the above adjoint definition corresponds to the conjugate-transpose for matrices.

In the subsequent parts of this problem, you may *not* assume that  $\hat{O}$  is finite-dimensional (nor may you assume any specific formula for the inner product).

- (b) Show that if  $\hat{O}$  is simply a number  $o$ , then  $\hat{O}^\dagger = o^*$ . (This is *not* the same as the previous question, since  $\hat{O}$  here can act on infinite-dimensional spaces.)
- (c) If a linear operator  $\hat{O}$  satisfies  $\hat{O}^\dagger = \hat{O}^{-1}$ , then the operator is called **unitary**. Show that a unitary operator preserves inner products (that is, if we apply  $\hat{O}$  to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues  $u$  of a unitary operator have unit magnitude ( $|u| = 1$ ) and that its eigenvectors can be chosen to be orthogonal to one another.
- (d) For a non-singular operator  $\hat{O}$  (i.e.  $\hat{O}^{-1}$  exists), show that  $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$ . (Thus, if  $\hat{O}$  is Hermitian then  $\hat{O}^{-1}$  is also Hermitian.)

### Problem 2: Completeness

- (a) Prove that the eigenvectors  $H_n$  of a *finite-dimensional* Hermitian operator  $\hat{O}$  (a  $d \times d$  matrix) are *complete*: that is, that any  $d$ -dimensional vector can be expanded as a sum  $\sum_n c_n H_n$  in the eigenvectors  $H_n$  with some coefficients  $c_n$ . It is sufficient to show that there are  $d$  linearly independent eigenvectors  $H_n$ :

- (i) Show that every  $d \times d$  Hermitian matrix  $O$  has at least one nonzero eigenvector  $H_1$  [... use the fundamental theorem of algebra: every polynomial with degree  $> 0$  has at least one (possibly complex) root].
- (ii) Show that the space of  $V_1 = \{H \mid \langle H, H_1 \rangle = 0\}$  orthogonal to  $H_1$  is preserved (transformed into itself or a subset of itself) by  $\hat{O}$ . From this, show that we can form a  $(d-1) \times (d-1)$  Hermitian matrix whose eigenvectors (if any) give (via a similarity transformation) the remaining (if any) eigenvectors of  $\hat{O}$ .
- (iii) By induction, form an orthonormal basis of  $d$  eigenvectors for the  $d$ -dimensional space.

- (b) Completeness is not automatic for eigenvectors in general. Give an example of a non-singular *non-Hermitian* operator whose eigenvectors are *not* complete. (A  $2 \times 2$  matrix is fine. This case is also called “defective.”)
- (c) Completeness of the eigenfunctions is not automatic for Hermitian operators on infinite-dimensional spaces either; they need to have some additional properties (e.g. “compactness”) for this to be true. However, it is true of most operators that we encounter in physical problems. If a particular operator did *not* have a complete basis of eigenfunctions, what would this mean about our ability to simulate the solutions on a computer in a finite computational box (where, when you discretize the problem, it turns approximately into a finite-dimensional problem)? No rigorous arguments required here, just your thoughts.

### Problem 3: Maxwell eigenproblems

- (a) In class, we eliminated  $\mathbf{E}$  from Maxwell’s equations to get an eigenproblem in  $\mathbf{H}$  alone, of the form  $\hat{O}\mathbf{H}(\mathbf{x}) = \frac{\omega^2}{c^2}\mathbf{H}(\mathbf{x})$ . Show that if you instead eliminate  $\mathbf{H}$ , you *cannot* get a Hermitian eigenproblem in  $\mathbf{E}$  except for the trivial case  $\varepsilon = \text{constant}$ . Instead, show that you get a *generalized Hermitian eigenproblem*: an equation of the form  $\hat{A}\mathbf{E}(\mathbf{x}) = \frac{\omega^2}{c^2}\hat{B}\mathbf{E}(\mathbf{x})$ , where *both*  $\hat{A}$  and  $\hat{B}$  are Hermitian operators.

- (b) For *any* generalized Hermitian eigenproblem where  $\hat{B}$  is positive definite (i.e.  $\langle \mathbf{E}, \hat{B}\mathbf{E} \rangle > 0$  for all  $\mathbf{E}(\mathbf{x}) \neq 0$ <sup>1</sup>), show that the eigenvalues (i.e., the solutions of  $\hat{A}\mathbf{E} = \lambda\hat{B}\mathbf{E}$ ) are real and that different eigenfunctions  $\mathbf{E}_1$  and  $\mathbf{E}_2$  satisfy a modified kind of orthogonality. Show that  $\hat{B}$  for the  $\mathbf{E}$  eigenproblem above was indeed positive definite.
- (c) Show that *both* the  $\mathbf{E}$  and  $\mathbf{H}$  formulations lead to generalized Hermitian eigenproblems with real  $\omega$  if we allow magnetic materials  $\mu(\mathbf{x}) \neq 1$  (but require  $\mu$  real, positive, and independent of  $\mathbf{H}$  or  $\omega$ ).
- (d)  $\mu$  and  $\varepsilon$  are only ordinary numbers for *isotropic* media. More generally, they are  $3 \times 3$  matrices (technically, rank 2 tensors)—thus, in an *anisotropic medium*, by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in  $\mathbf{E}$  (or  $\mathbf{H}$ ) with real eigenfrequency  $\omega$ .

---

<sup>1</sup>Here, when we say  $\mathbf{E}(\mathbf{x}) \neq 0$  we mean it in the sense of generalized functions; loosely, we ignore isolated points where  $\mathbf{E}$  is nonzero, as long as such points have zero integral, since such isolated values are not physically observable. See e.g. Gelfand and Shilov, *Generalized Functions*.