18.369 Midterm Exam (Spring 2010)

April 7, 2010

You have two hours.

Problem 1: Irreps (10 pts)

Suppose you are given the character table $\chi^{(\alpha)}(g)$ for some group G for all $g \in G$ and all irreducible representations α. Now, someone gives you a representation $D(g)$ of G, which is unitary but *may* be reducible.

Given *only* the trace $\chi(g) = \operatorname{trace} D(g)$ of this representation (for all $g \in G$), explain how to figure out which irreducible representations $D(g)$ decomposes into.

Problem 2: Surface states (20 pts)

Suppose that you have a semi-infinite multilayer film (1d crystal) for $x > 0$ in two dimensions, with air for $x < 0$ as depicted in Fig. 1(top), with the terminated so as to have a surface state at the edge of the crystal, with the TM projected band diagram in Fig. 1(bottom). The structure is infinite in the y direction (your lab has a lot of space).

(a) One of the post-docs in your group, Pat Ern, claims to have excited the surface state by pointing a laser (somewhere in the air, $x < 0$) at the surface. Your professor is skeptical—why? Assuming Pat is telling the truth, how might this have been achieved?

(b) Your professor suggests that the prism structure in Fig. 2 might be an easier way of exciting the surface state from an external laser. How can this work? Is there any condition that ε_p (the relative permittivity of the prism) needs to satisfy for a given laser frequency ω? What about the angle θ of the prism, assuming the laser is incident normal to the prism facet as shown?

Problem 3: 2d photonic crystal (20 pts)

The symmetry operations for the regular hexagon (symmetry group C_{6v}) and the equilateral triangle (symmetry group C_{3v}, from problem set 2) are shown in figure 3. The corresponding character tables are:

C_{6v}	E	$2C_6$	$2C_3$	C_2	3σ	$3\sigma'$
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1
Γ_3	1	-1	1	-1	1	-1
Γ_4	1	-1	1	-1	-1	1
Γ_5	2	-1	-1	2	0	0
Γ_6	2	1	-1	-2	0	0

C_{3v}	E	$2C_3$	3σ
R_1	1	1	1
R_2	1	1	-1
R_3	2	-1	0

(where we have called the C_{6v} representations $\Gamma_{1,\ldots,6}$ and the C_{3v} representations $R_{1,\ldots,6}$ to distinguish them). C_{6v} is the symmetry group of a triangular lattice of circular dielectric rods ($\varepsilon = 12$) in air ($\varepsilon = 1$), shown as an inset in Fig. 4, along with its TM band structure in Fig. 4 around the edges of the irreducible Brillouin zone (right inset). The primitive lattice vectors are $\mathbf{R}_1 = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}\sqrt{3})$ and $\mathbf{R}_2 = \frac{a}{2}(\hat{\mathbf{x}} - \hat{\mathbf{y}}\sqrt{3})$, and the corresponding primitive reciprocal lattice vectors are $\mathbf{G}_1 = \frac{2\pi}{a}(\hat{\mathbf{x}} + \hat{\mathbf{y}}/\sqrt{3})$ and $\mathbf{G}_2 = \frac{2\pi}{a}(\hat{\mathbf{x}} - \hat{\mathbf{y}}/\sqrt{3})$. In this basis, we can write $M = \mathbf{G}_1/2$ and $K = (\mathbf{G}_1 + \mathbf{G}_2)/3$.

(a) Give the symmetry groups (just rotations/reflections, no translations) of $\hat{\Theta}_{\mathbf{k}}$ at $\mathbf{k} = \Gamma$, $\mathbf{k} = M$, $\mathbf{k} = K$, and $\mathbf{k} \in (M, K)$ (i.e., the line segment from M to K, not including the endpoints). For what $\mathbf{k}$ would you expect non-accidental degeneracies to occur, and why?

(b) Sketch what you expect for the E_z field patterns for the lowest two ω at K (note that the second eigenvalue is doubly degenerate) and for the first three eigenvalues at M. You can just sketch positive and negative fields as "+" and "−" like I do in class. Sketch the fields at least a few periods of the crystal (e.g. the 4×4 set of periods in the inset of Fig. 4) so that the phase relationships between unit cells are clear—in particular, be sure to indicate for which cylinders the field pattern is merely multiplied by ± 1 compared to a cylinder at the origin.
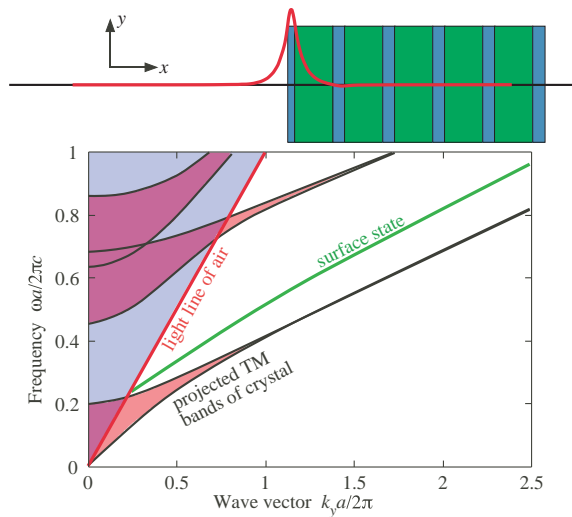
Figure 1: Top: A semi-infinite multilayer film to the right, with air to the left, terminated so as to have a surface state (red curve) for the TM polarization (E_z). Bottom: TM projected band diagram, showing projection of crystal modes, light cone of air, and the surface state.
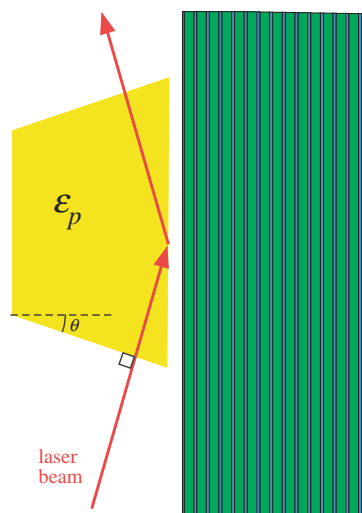

Figure 2: Proposed prism structure for coupling a (TM) laser beam into a surface state of the multilayer film.
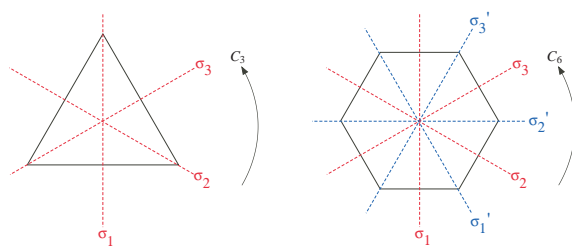

Figure 3: Left: symmetry operations (C_{3v}) for the triangle (three mirror planes and 3-fold rotations). Right: symmetry operations (C_{3v}) for the hexagon (six mirror planes and 6-fold rotations).
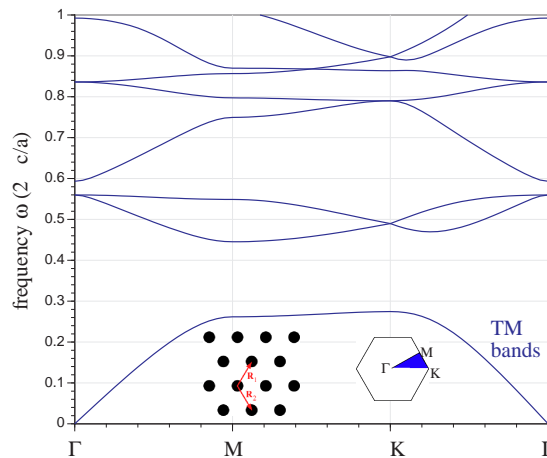

Figure 4: TM structure of triangular lattice (period a) of dielectric ($\varepsilon = 12$) cylinders (radius $0.2a$) in air. Insets show the structure and the irreducible Brillouin zone.