18.369:
Mathematical Methods in Nanophotonics

overview lecture slides

(don't get used to it: most lectures are blackboard)

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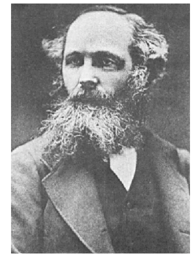
MIT Applied Mathematics

Maxwell's Equations



James Clerk Maxwell 1864

Gauss: $\nabla \cdot \mathbf{B} = 0$

$\nabla \cdot \mathbf{D} = \rho$

Ampere: $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

Faraday: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

constitutive relations:

$\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$

$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$

electromagnetic fields:

E = electric field
D = displacement field
H = magnetic field / induction
B = magnetic field / flux density

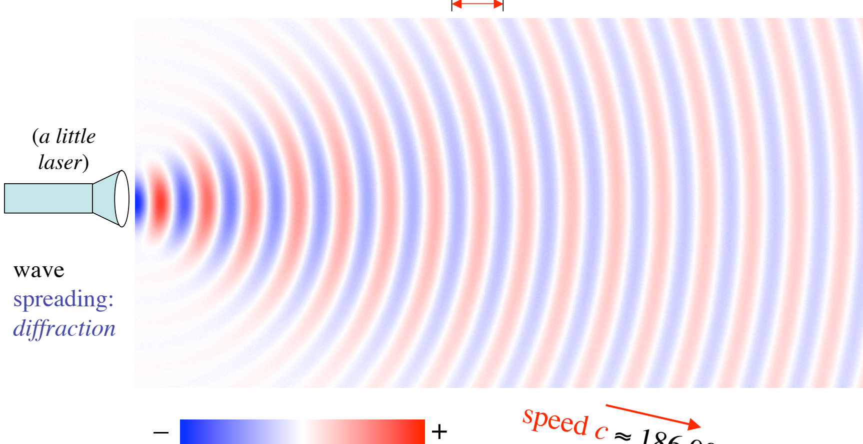
sources: **J** = current density
ρ = charge density

constants: ε_0, μ_0 = vacuum permittivity/permeability
c = vacuum speed of light = $(\varepsilon_0\, \mu_0)^{-1/2}$

material response to fields:
P = polarization density
M = magnetization density

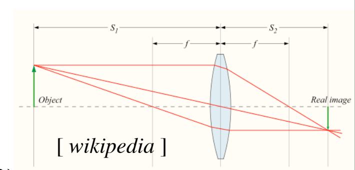
Light is a Wave (and a quantum particle)

color ⇔ wavelength $\lambda \approx \frac{1}{2000}$ mm for visible light



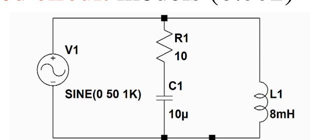
When can we solve this mess?

- Very small wavelengths: ray optics



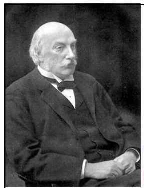
[*wikipedia*]

- Very large wavelengths:
  quasistatics (8.02)
  & lumped circuit models (6.002)



[homestead.com]

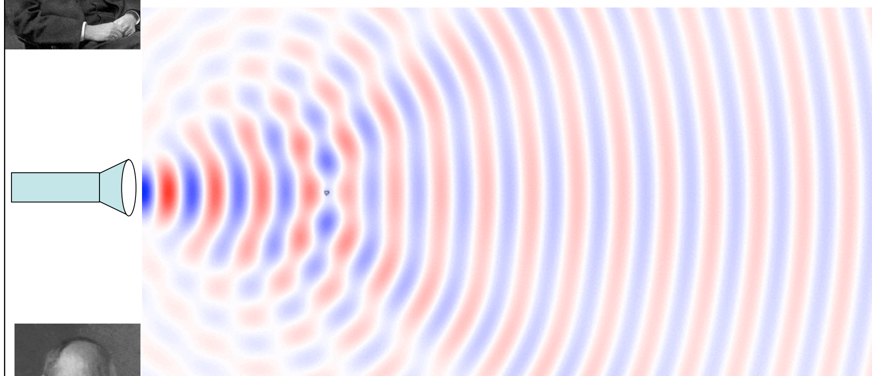
- Wavelengths comparable to geometry?
  - handful of cases can be ~solved analytically: planes, spheres, cylinders, empty space (8.07, 8.311)
  - everything else just a mess for computer…?



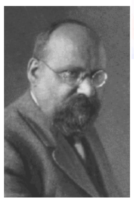
small particles:
Lord Rayleigh (1871)
why the sky is blue

Waves Can Scatter

here: a little circular speck of silicon



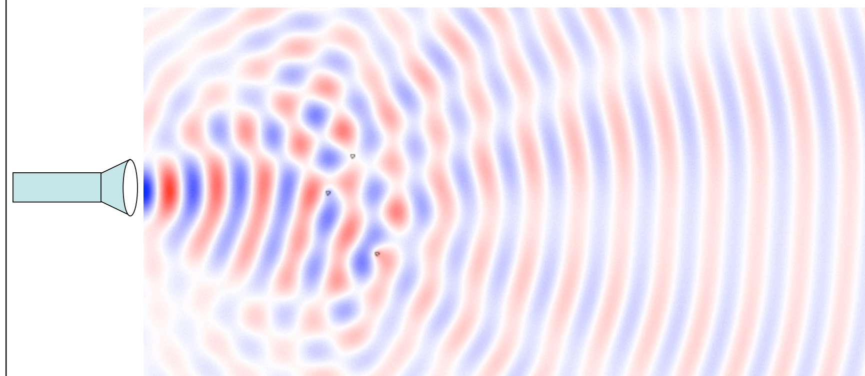
checkerboard pattern: interference of waves
traveling in different directions



scattering by spheres:
solved by Gustave Mie (1908)

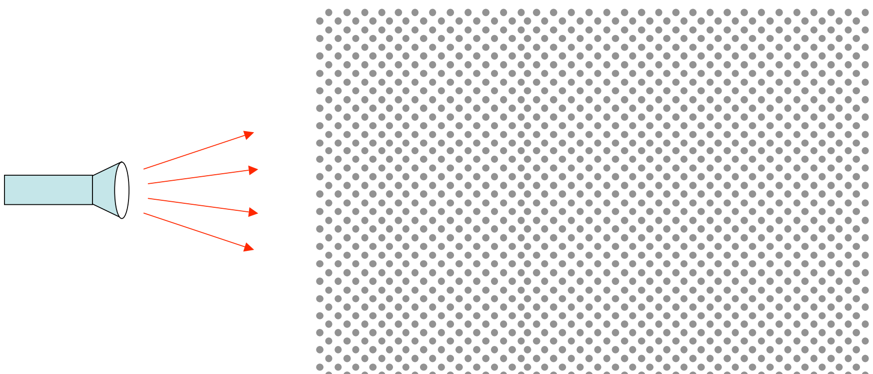
Multiple Scattering is Just Messier?

here: scattering off three specks of silicon



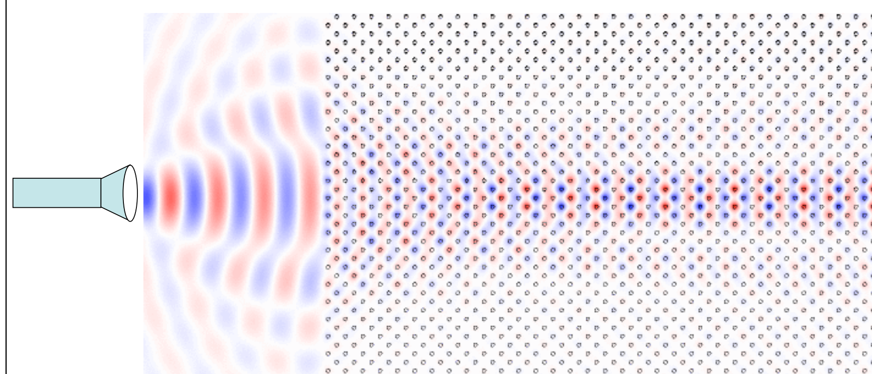
can be solved on a computer, but not terribly interesting…

An even bigger mess?
zillons of scatterers



Blech, light will just scatter like crazy
and go all over the place … how boring!

Not so messy, not so boring…



the light seems to form several *coherent beams*
that propagate *without scattering*
… and almost *without diffraction* (*supercollimation*)

…the magic of symmetry…

[Emmy Noether, 1915]

Noether's theorem:
symmetry = conservation laws

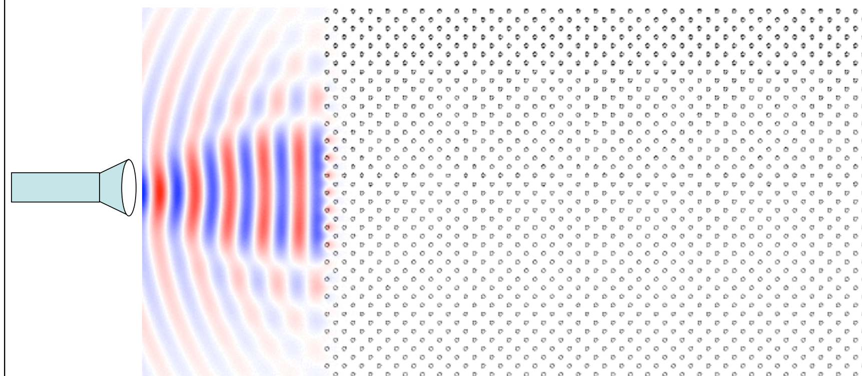
In this case, periodicity
= conserved "momentum"
= wave solutions without scattering
[Bloch waves]

Felix Bloch
(1928)

Mathematically, use *structure* of the equations, not explicit solution:
linear algebra, group theory, functional analysis, …

A slight change? Shrink λ by 20%

an "optical insulator" (photonic bandgap)



light cannot penetrate the structure at this wavelength!
all of the scattering destructively interferes

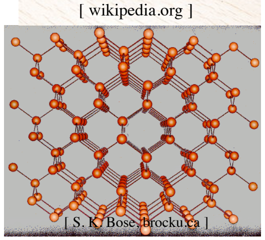
Photonic Crystals:

periodic structures for light

crystalline minerals:
periodic atoms

silicon:

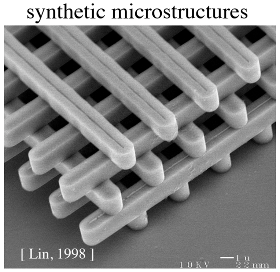
[wikipedia.org]



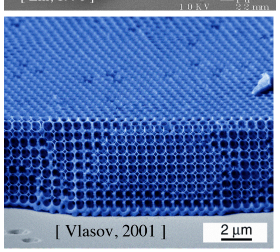
[S. K. Bose, brocku.ca]

bandgaps = insulators & semiconductors

synthetic microstructures



[Lin, 1998]



[Vlasov, 2001]

Lots of math to *understand* this…

…linear algebra:

$$\nabla \times \frac{1}{\varepsilon} \nabla \times \vec{H} = \left(\frac{\omega}{c}\right)^2 \vec{H}$$

(eigenproblem)

…symmetry ⇒ group representation theory

…computational methods…

…many open questions…

Structural Color in Nature

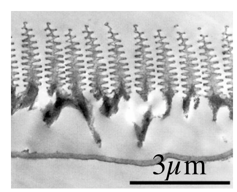
bandgap = wavelength-selective mirror = bright iridescent colors

opals

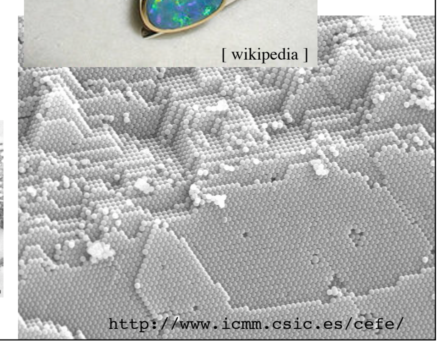
[wikipedia]

wing scale:

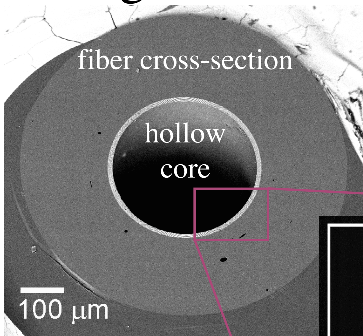
[P. Vukosic (1999)]

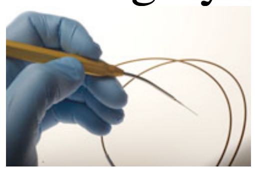


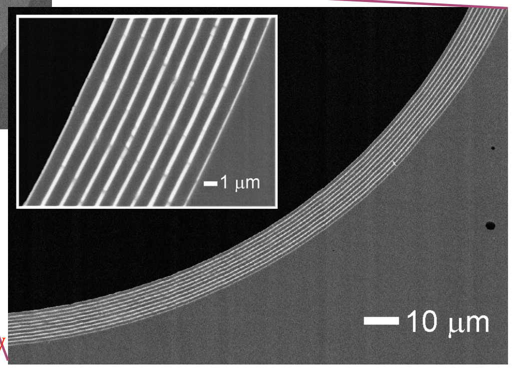
also peacocks, beetles, …



Light Confinement for Surgery




MIT startup: omni-guide.com

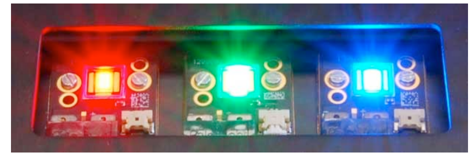


Periodic layering
confines laser light
that would vaporize glass
within a hollow core

Used for endoscopic surgery

Molding Diffraction for Lighting

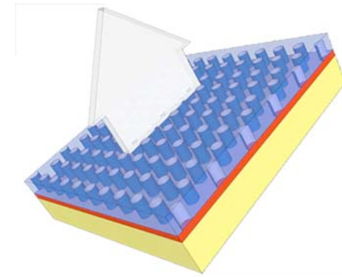
[another MIT startup (by a colleague): Luminus.com]



ultra-bright/efficient LEDs

periodic pattern gathers &
redirects it in one direction

new projection TVs,
pocket projectors,
lighting applications,
…



Back to Maxwell, with some simplifications

- *source-free* equations (propagation of light, not creation): $\mathbf{J} = 0$, $\rho = 0$

- *Linear*, *dispersionless* (instantaneous response) materials:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \Rightarrow \quad \mathbf{D} = \varepsilon_0 (1+\chi_e) \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad\quad \mathbf{B} = \mu_0 (1+\chi_m) \mathbf{H} = \mu_0 \mu_r \mathbf{H}$$

(nonlinearities very weak in EM … we'll treat later)
(dispersion can be negligible in narrow enough bandwidth)

where $\varepsilon_r = 1+\chi_e$ = relative permittivity or dielectric constant
(drop r subscript)
$\mu_r = 1+\chi_m$ = relative perme

$\varepsilon\mu$ = (refractive index)^2

- *Isotropic* materials: ε, μ = scalars (not matrices)
- *Non-magnetic* materials: $\mu = 1$ (true at optical/infrared)
- *Lossless, transparent materials*: ε real, > 0 (< 0 for metals…bad at infrared)

Simplified Maxwell

$$\nabla \cdot \mathbf{H} = 0 \qquad \nabla \cdot \varepsilon\mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \varepsilon(\mathbf{x}) \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\mu_0 \mu \frac{\partial \mathbf{H}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

- Linear, time-invariant system:
  $\Rightarrow$ look for sinusoidal solutions $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$, $\mathbf{H}(\mathbf{x},t) = \mathbf{H}(\mathbf{x})e^{-i\omega t}$

  (i.e. Fourier transform)

$$\nabla \times \mathbf{H} = -i\omega\varepsilon_0 \varepsilon(\mathbf{x})\mathbf{E} \qquad \nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}$$

[note: real materials have *dispersion*: ε depends on ω = *non-instantaneous* response]

… these, we can work with