18.369 Midterm Exam (Spring 2009)

March 31, 2010

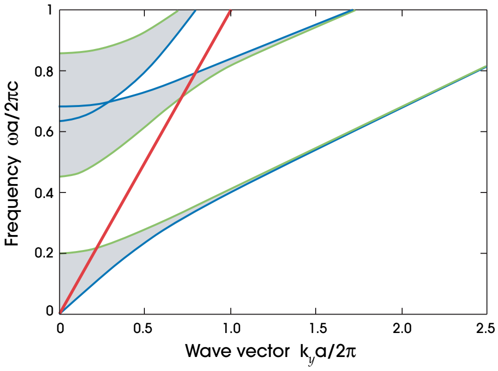

Figure 1: Projected band diagram of periodic multilayer film for problem 1, *without* defects, for the TM polarization and a wavevector k_y parallel to the layers. (Solid red line is the light line of air.)

You have two hours. **There are three problems, each worth 30 points.**

Problem 1: Projected bands

In figure 1 is shown the projected band diagram of a multilayer film (a quarter-wave stack with period a for alternating layers of $\varepsilon = 13$ and $\varepsilon = 1$) for a wavevector k_y parallel to the layers. This is for the TM polarization (**E** perpendicular to k_y and parallel to the layers). Now, suppose that we introduce a defect: a sequence of circular air holes, with period $\Lambda = a$, in the middle of *one* of the high-index layers, as shown in figure 2.

(a) Assume that the defect structure of figure 2 introduces a guided mode. Sketch what you think is the band diagram for the most likely guided mode, and also the field pattern of the guided mode at $k_y = 0$. (If you think there might be more than one guided mode, just sketch the lowest-ω mode.)

(b) Suppose you increase the period Λ from one hole to the next. For any Λ,should it always be possible to create a guided mode by adding air holes (possibly of a different shape), or is there a maximum Λ beyond which air holes will never introduce a guided mode? Explain your reasoning. (No equations are needed, but a sketch or two might be nice.)

Problem 2: Perturbations

Suppose that we have a electromagnetic system with both a dielectric permittivity $\varepsilon > 0$ and a magnetic permeability $\mu > 0$. As you

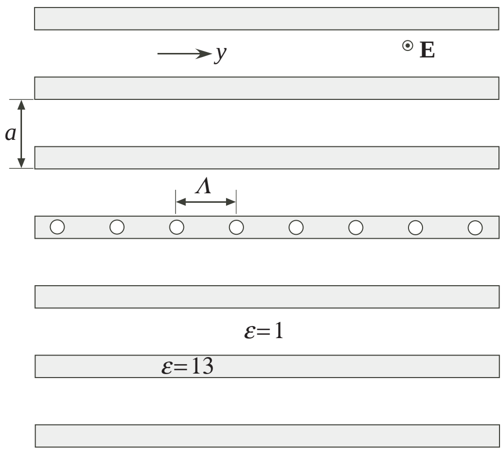

Figure 2: Multilayer film with defect layer (air holes) for problem 1.

found in homework, this is described by the Hermitian *generalized* eigenproblem:

$$\nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} = \frac{\omega^2}{c^2} \mu \mathbf{H}. \tag{1}$$

Furthermore, however, suppose $\varepsilon(\omega)$ and $\mu(\omega)$ are (real, positive, smooth [with Taylor expansions]) functions of the frequency ω (that is, a weakly dispersive medium). In this case, we have a "nonlinear eigenproblem" (1), not simply a linear generalized eigenproblem (it is nonlinear in ω, not in **H**). But the operators are still Hermitian etcetera at any given ω (as usual, assume that **H** goes to zero sufficiently far away, or is bounded in some other way, so that $\langle \mathbf{H}, \mathbf{H} \rangle$ exists and boundary terms are irrelevant in integrals).

(a) Explain why, if equation (1) has a solution ω and **H**, then ω must be real. (Note that you need not rederive things that were proved in class, such as the fact that $\nabla\times$ is Hermitian. But you cannot simply quote results we proved for linear eigenproblems without clearly explaining why they apply here.)

(b) Suppose you are given a nondegenerate solution ω and **H** to equation (1). If we change $\varepsilon(\omega)$ to $\varepsilon(\omega) + \Delta\varepsilon$ and $\mu(\omega)$ to $\mu(\omega) + \Delta\mu$, find the first-order correction (in $\Delta\varepsilon$ and $\Delta\mu$) to ω. Explain why any smooth frequency dependence of $\Delta\varepsilon$ and $\Delta\mu$ is irrelevant to this first-order calculation; all we need to know are $\Delta\varepsilon(\omega)$ and $\Delta\mu(\omega)$. (The frequency dependence of ε and μ is *not* irrelevant: be careful not to discard any first-order terms!)

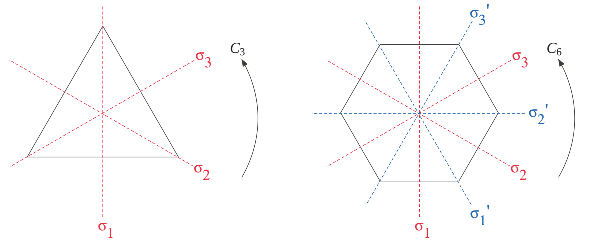

Figure 3: Left: symmetry operations (C_{3v}) for the triangle (three mirror planes and 3-fold rotations). Right: symmetry operations (C_{3v}) for the hexagon (six mirror planes and 6-fold rotations).

Problem 3: Symmetries

The symmetry operations for the regular hexagon (symmetry group C_{6v}, from class and the 2007 midterm) and the equilateral triangle (symmetry group C_{3v}, from problem set 2) are shown in figure 3. The corresponding character tables are:

C_{6v}	E	$2C_6$	$2C_3$	C_2	3σ	$3\sigma'$
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1
Γ_3	1	-1	1	-1	1	-1
Γ_4	1	-1	1	-1	-1	1
Γ_5	2	-1	-1	2	0	0
Γ_6	2	1	-1	-2	0	0

C_{3v}	E	$2C_3$	3σ
R_1	1	1	1
R_2	1	1	-1
R_3	2	-1	0

(where we have called the C_{6v} representations $\Gamma_{1,\ldots,6}$ and the C_{3v} representations $R_{1,\ldots,6}$ to distinguish them).

(a) Suppose you start with an air-filled hexagonal metallic-walled cavity, in two dimensions, and compute eigenmodes corresponding to all of the six irreducible representations ($\Gamma_{1,\ldots,6}$), with no accidental degeneracies. Now we put a concentric *triangle* of some perturbation $\Delta\varepsilon$ inside the hexagon, where the triangle is concentric with the outer hexagon and the bottom edge of the triangle is parallel with the bottom edge of the hexagon. For each of $\Gamma_{1,\ldots,6}$, describe whether eigenfunctions of that representation in the original structure correspond in the perturbed structure to partner functions of reducible or irreducible representations of the perturbed structure—and in the latter case, which of $R_{1,\ldots,3}$ they correspond to in the new symmetry group C_{3v}.

(b) Suppose you are doing a Meep simulation of the fields in this hexagonal cavity with the triangular inclusion. You want to put in current sources **J** that excite modes of each of the irreducible representations $R_{1,\ldots,3}$ one at a time. That is, first you want to put in a **J** that excites only R_1 to study the R_1 modes (in response to a short pulse), then a current for only R_2, then a current for only R_3. Explicitly give examples of such currents. You need give only the spatial pattern $\mathbf{J}(\mathbf{x})$. (Note that the simplest possible current is a point dipole, where **J** is a delta function at some point. I would suggest using one or more point dipoles; be sure to indicate both their locations and the orientation of **J** at each dipole location. A drawn picture is sufficient.)

(c) Suppose you are given a field pattern of an eigenmode of the hexagonal cavity with the triangular $\Delta\varepsilon$. You know that the field pattern must be a partner function of an irreducible representation, but you're not sure which one. Explain how (and why!) you can determine which irreducible representation the field is by evaluating the field at only six points in space, as long as the field does not happen to be zero at those points. (*Hint*: start with the projection operator on the field at all points, and then show how you can simplify the problem.)