18.369 Problem Set 2

Due Friday, 29 February 2008.

Problem 1: Projection operators (easy)

The representation-theory handout gives a formula for the projection operator from a state onto its component that transforms as a particular representation. Prove the correctness of this formula (using the Great Orthogonality Theorem).

Problem 2: Symmetries of a field in a square metal box

In class, we considered a two-dimensional (xy) problem of light in an $L \times L$ square of air ($\varepsilon = 1$) surrounded by perfectly conducting walls (in which $\mathbf{E} = 0$). We solved the case of $\mathbf{H} = H_z(x, y)\hat{\mathbf{z}}$ and saw solutions corresponding to five different representations of the symmetry group ($C_{4\mathrm{V}}$).

(a) Solve for the eigenmodes of the other polarization: $\mathbf{E} = E_z(x, y)\hat{\mathbf{z}}$ (you will need the $\mathbf{E}$ eigenproblem from problem set 1), with the boundary condition that $E_z = 0$ at the metal walls.

(b) Sketch and classify these solutions according to the representations of $C_{4\mathrm{V}}$ enumerated in class. (Like in class, you will get some reducible accidental degeneracies.)

Problem 3: Symmetries of a field in a triangular metal box

Consider the two-dimensional solutions in a *triangular* perfect-metal box with side L. Don't try to solve this analytically; instead, you will use symmetry to sketch out what the possible solutions will look like for both E_z and H_z polarizations.

(a) List the symmetry operations in the space group (choose the origin at the center of the triangle so that the space group is symmorphic), and break them into conjugacy classes. (This group is traditionally called $C_{3\mathrm{V}}$). Verify that the group is closed under composition (i.e. that the composition of two operations always gives another operation in the group) by giving the "multiplication table" of the group (whose rows and columns are group members and whose entries give their composition).

(b) Find the character table of $C_{3\mathrm{V}}$, using the rules from the representation-theory handout.

(c) Give unitary representation matrices D for each irreducible representation of $C_{3\mathrm{V}}$.

(d) Sketch possible $\omega \neq 0$ E_z and H_z solutions that would transform as these representations. What representation should the lowest-ω mode (excluding $\omega = 0$) of each polarization correspond to?

(e) If there are any (non-accidental) degenerate modes, show how given one of the modes we can get the other orthogonal eigenfunction(s) (e.g. in the square case we could get one from the 90° rotation of the other for a degenerate pair, but the triangular structure is not symmetric under 90° rotations). Hint: use your representation matrices.

Problem 4: Cylindrical symmetry

Suppose that we have a *cylindrical* metallic waveguide—that is, a perfect metallic tube with radius R, which is uniform in the z direction. The interior of the tube is simply air ($\varepsilon = 1$).

(a) This structure has continuous rotational symmetry around the z axis, called the C_∞ group.[1] Find the irreducible representations of this group (there are infinitely many because it is an infinite group).

(b) For simplicity, consider the (Hermitian) *scalar* wave equation $-\nabla^2\psi = \frac{\omega^2}{c^2}\psi$ with $\psi|_{r=R} = 0$. Show that, when we look for solutions ψ that transform like one of the representations of the

[1] It also has an infinite set of mirror planes containing the z axis, but let's ignore these for now. If they are included, the group is called $C_{\infty\mathrm{V}}$.

C_∞ group from above, and have z dependence e^{ikz} (from the translational symmetry), then we obtain a Bessel equation (Google it if you've forgotten Mr. Bessel). Write the solutions in terms of Bessel functions, assuming that you are given their zeros $x_{m,n}$ (i.e. $J_m(x_{m,n}) = 0$ for $n = 1, 2, \ldots$, where J_m is the Bessel function of the first kind...if you Google for "Bessel function zeros" you can find them tabulated). Sketch the dispersion relation $\omega(k)$ for a few bands.

(c) From the general orthogonality of Hermitian-operator eigenfunctions, derive/prove an orthogonality integral for the Bessel functions. (No, just looking one up on Wikipedia doesn't count.)

Problem 5: Conservation Laws

Suppose that we introduce a nonzero current $\mathbf{J}(\mathbf{x})e^{-i\omega t}$ into Maxwell's equations at a given frequency ω, and we want to find the resulting time-harmonic electric field $\mathbf{E}(\mathbf{x})e^{-i\omega t}$ (i.e. we are only looking for fields that arise from the current, with $\mathbf{E} \to 0$ as $|\mathbf{x}| \to \infty$ if $\mathbf{J}$ is localized).

(a) Show that this results in a linear equation of the form $\hat{A}\,|E\rangle = |b\rangle$, where $\hat{A}$ is some linear operator and $|b\rangle$ is some known right-hand side in terms of the current density $\mathbf{J}$.

(b) Prove that, if $\mathbf{J}$ transforms as some irreducible representation of the space group then $|E\rangle$ ($= \mathbf{E}$, which you can assume is a unique solution) does also. (This is the analogue of the conservation in *time* that we showed in class, except that now we are proving it in the frequency domain. You could prove it by Fourier-transforming the theorem from class, I suppose, but do *not* do so—instead, prove it directly from the linear equation here.)

(c) Formally, $|E\rangle = \hat{A}^{-1}\,|b\rangle$, where $\hat{A}^{-1}$ is related to the *Green's function* of the system. What happens if ω is one of the eigenfrequencies?