18.369 Problem Set 1

Due Friday, 16 February 2007.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of states and operators by: $\langle H_1 | H_2 \rangle = | H_1 \rangle \langle H_2 \rangle$ and $\langle H_1 | \hat{O} | H_2 \rangle = (\hat{O} | H_1 \rangle) \langle H_2 \rangle$. Show that for a finite-dimensional Hilbert space, where $| H \rangle$ is a column vector $h_n$ ($n = 1, \ldots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $(H^{(1)} | H^{(2)})$ is the ordinary conjugated dot product $\sum_n h_n^{(1)} h_n^{(2)}$, the above adjoint definition corresponds to the conjugate-transpose for both matrices and vectors.

(b) Show that if $\hat{O}$ is simply a number $a$, then $\hat{O}^\dagger = a^*$. (This is not the same as the previous question, since $\hat{O}$ here can act on infinite-dimensional (continuous) spaces.)

(c) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called unitary. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues $\omega$ of a unitary operator have unit magnitude ($|\omega| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(d) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian, then $\hat{O}^{-1}$ is also Hermitian.)

Problem 2: Completeness

(a) Prove that the eigenvectors $|n\rangle$ of a finite-dimensional Hermitian operator $\hat{O}$ (a $d \times d$ matrix) are complete: that is, that any $d$-dimensional vector $c_n |n\rangle$ in the eigenvectors with some coefficients $c_n$. It is sufficient to show that there are $d$ linearly independent eigenvectors $|n\rangle$:

(i) Show that every $d \times d$ Hermitian matrix $\hat{O}$ has at least one nonzero eigenvector $|1\rangle$ (... use the fact that every polynomial with nonzero degree has at least one (possibly complex) root).

(ii) Show that the space of $V_1 = \{ |v\rangle | \langle v|1\rangle = 0 \}$ orthogonal to $|1\rangle$ is preserved (transformed into itself or a subset of itself) by $\hat{O}$. From this, show that we can form a $(d - 1) \times (d - 1)$ Hermitian matrix whose eigenvectors (if any) give (via a similarity transformation) the remaining (if any) eigenvectors of $\hat{O}$.

(iii) By induction, form an orthonormal basis of $d$ eigenvectors for the $d$-dimensional space.

(b) Completeness is not automatic for eigenvectors in general. Give an example of a non-singular non-Hermitian operator whose eigenvectors are not complete. (A $2 \times 2$ matrix is fine. This case is also called “defective.”)

Problem 3: Maxwell eigenproblems

(a) In class, we eliminated $E$ from Maxwell’s equations to get an eigenproblem in $H$ alone, of the form $\hat{\Theta} |H\rangle = \frac{\omega^2}{c^2} |H\rangle$. Show that if you instead eliminate $H$, you cannot get a Hermitian eigenproblem in $E$ except for the trivial case $\varepsilon = \text{constant}$. Instead, show that you get a generalized Hermitian eigenproblem of the form $\hat{\Theta} |E\rangle = \frac{\omega^2}{c^2} \hat{B} |E\rangle$, where both $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For any generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle E | \hat{B} | E \rangle > 0$ for all $|E\rangle \neq 0$), show that the eigenvalues are real and that different eigenvectors $|E_1\rangle$ and $|E_2\rangle$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $E$ eigenproblem above was indeed positive definite.

(c) Show that both the $|E\rangle$ and $|H\rangle$ formulations lead to generalized Hermitian eigenproblems.

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1Here, when we say $|E\rangle \neq 0$ we mean it in the sense of generalized functions; loosely, we ignore isolated points where $E$ is nonzero, as long as such points have zero integral, since such isolated values are not physically observable. See e.g. Gelfand and Shilov, Generalized Functions.
with real $\omega$ if we allow magnetic materials $\mu(x) \neq 1$ (but require $\mu$ real, positive, and independent of $\mathbf{H}$ or $\omega$).

(d) $\mu$ and $\epsilon$ are only ordinary numbers for isotropic media. More generally, they are $3 \times 3$ matrices (technically, rank 2 tensors)—thus, in an anisotropic medium, by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in $\mathbf{E}$ (or $\mathbf{H}$) with real eigen-frequency $\omega$. 