18.369 Problem Set 1

Due Friday, 16 February 2007.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of states and operators by: $\langle H_1|H_2\rangle = |H_1\rangle^\dagger |H_2\rangle$ and $\langle H_1|\hat{O}|H_2\rangle = (\hat{O}^\dagger |H_1\rangle)^\dagger |H_2\rangle$. Show that for a *finite-dimensional* Hilbert space, where $|H\rangle$ is a column vector h_n ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}|H^{(2)}\rangle$ is the ordinary conjugated dot product $\sum_n h_n^{(1)*} h_n^{(2)}$, the above adjoint definition corresponds to the conjugate-transpose for both matrices and vectors.

(b) Show that if $\hat{O}$ is simply a number o, then $\hat{O}^\dagger = o^*$. (This is not the same as the previous question, since $\hat{O}$ here can act on infinite-dimensional (continuous) spaces.)

(c) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called **unitary**. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues u of a unitary operator have unit magnitude ($|u| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(d) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

Problem 2: Completeness

(a) Prove that the eigenvectors $|n\rangle$ of a *finite-dimensional* Hermitian operator $\hat{O}$ (a $d \times d$ matrix) are *complete*: that is, that any d-dimensional vector can be expanded as a sum $\sum_n c_n |n\rangle$ in the eigenvectors with some coefficients c_n. It is sufficient to show that there are d linearly independent eigenvectors $|n\rangle$:

   (i) Show that every $d \times d$ Hermitian matrix O has at least one nonzero eigenvector $|1\rangle$ (... use the fact that every polynomial with nonzero degree has at least one (possibly complex) root).

   (ii) Show that the space of $V_1 = \{|v\rangle \,|\, \langle v|1\rangle = 0\}$ orthogonal to $|1\rangle$ is preserved (transformed into itself or a subset of itself) by $\hat{O}$. From this, show that we can form a $(d-1) \times (d-1)$ Hermitian matrix whose eigenvectors (if any) give (via a similarity transformation) the remaining (if any) eigenvectors of O.

   (iii) By induction, form an orthonormal basis of d eigenvectors for the d-dimensional space.

(b) Completeness is not automatic for eigenvectors in general. Give an example of a non-singular *non-Hermitian* operator whose eigenvectors are *not* complete. (A 2×2 matrix is fine. This case is also called "defective.")

Problem 3: Maxwell eigenproblems

(a) In class, we eliminated $\mathbf{E}$ from Maxwell's equations to get an eigenproblem in $\mathbf{H}$ alone, of the form $\hat{\Theta} |H\rangle = \frac{\omega^2}{c^2} |H\rangle$. Show that if you instead eliminate $\mathbf{H}$, you *cannot* get a Hermitian eigenproblem in $\mathbf{E}$ except for the trivial case $\varepsilon =$ constant. Instead, show that you get a *generalized Hermitian eigenproblem* of the form $\hat{A}|E\rangle = \frac{\omega^2}{c^2}\hat{B}|E\rangle$, where *both* $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For *any* generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle E|\hat{B}|E\rangle > 0$ for all $|E\rangle \neq 0$[1]), show that the eigenvalues are real and that different eigenvectors $|E_1\rangle$ and $|E_2\rangle$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $\mathbf{E}$ eigenproblem above was indeed positive definite.

(c) Show that *both* the $|E\rangle$ and $|H\rangle$ formulations lead to generalized Hermitian eigenproblems

[1] Here, when we say $|E\rangle \neq 0$ we mean it in the sense of generalized functions; loosely, we ignore isolated points where $\mathbf{E}$ is nonzero, as long as such points have zero integral, since such isolated values are not physically observable. See e.g. Gelfand and Shilov, *Generalized Functions*.

with real ω if we allow magnetic materials $\mu(\mathbf{x}) \neq 1$ (but require μ real, positive, and independent of $\mathbf{H}$ or ω).

(d) μ and ϵ are only ordinary numbers for *isotropic* media. More generally, they are 3×3 matrices (technically, rank 2 tensors)—thus, in an *anisotropic medium,* by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in $\mathbf{E}$ (or $\mathbf{H}$) with real eigen-frequency ω.