Coordinate Transformation & Invariance in Electromagnetism

Steven G. Johnson, notes for the course 18.369 at MIT

August 17, 2007

It is a remarkable fact [1] that Maxwell's equations under *any* coordinate transformation can be written in an *identical* "Cartesian" form, if simple transformations are applied to the materials (ε and μ), the fields ($\mathbf{E}$ and $\mathbf{H}$), and the sources (ρ and $\mathbf{J}$). This result has numerous useful and/or beautiful consequences, from designs of "invisibility cloaks" [2], to a simple derivation of PML absorbing boundaries [3], to enabling analyses of bent and twisted waveguides in terms analogous to a quantum Stark effect [4] , to providing a simple way of applying numerical methods designed for Cartesian coordinates to other coordinate systems [1].

Here, we review the proof in a compact form, generalized to arbitrary anisotropic media. (Most previous derivations seem to have been for isotropic media in at least one coordinate frame [1], or for coordinate transformations with purely diagonal Jacobians $\mathcal{J}$ where $\mathcal{J}_{ii}$ depends only on x_i [3], or for constant affine coordinate transforms [5].)

Summary of the Result

Maxwell's equations in Cartesian coordinates $\mathbf{x}$ are written (in natural units $\varepsilon_0 = \mu_0 = 1$):

$$\begin{aligned}
\nabla \times \mathbf{H} &= \boldsymbol{\varepsilon}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} && (1)\\
\nabla \times \mathbf{E} &= -\boldsymbol{\mu}\frac{\partial \mathbf{H}}{\partial t} && (2)\\
\nabla \cdot (\boldsymbol{\varepsilon}\mathbf{E}) &= \rho && (3)\\
\nabla \cdot (\boldsymbol{\mu}\mathbf{H}) &= 0, && (4)
\end{aligned}$$

where $\mathbf{J}$ and ρ are the usual free current and charge densities, respectively, and $\boldsymbol{\varepsilon}(\mathbf{x})$ and $\boldsymbol{\mu}(\mathbf{x})$ are the 3×3 relative permittivity and permeability tensors, respectively. Now, suppose that we make some (differentiable) coordinate transformation $\mathbf{x} \mapsto \mathbf{x}'$ (usually chosen to be non-singular, with some exceptions [2]). Let $\mathcal{J}$ denote the 3×3 Jacobian matrix:

$$\mathcal{J}_{ij} = \frac{\partial x'_i}{\partial x_j}.$$

We will show that Maxwell's equations take on the *same* form (1–4) in the primed coordinate system, with ∇ replaced by ∇', if we make the transformations:

$$\mathbf{E}' = (\mathcal{J}^T)^{-1}\mathbf{E}, \tag{5}$$

$$\mathbf{H}' = (\mathcal{J}^T)^{-1}\mathbf{H}, \tag{6}$$

$$\boldsymbol{\varepsilon}' = \frac{\mathcal{J}\boldsymbol{\varepsilon}\mathcal{J}^T}{\det \mathcal{J}}, \tag{7}$$

$$\boldsymbol{\mu}' = \frac{\mathcal{J}\boldsymbol{\mu}\mathcal{J}^T}{\det \mathcal{J}}, \tag{8}$$

$$\mathbf{J}' = \frac{\mathcal{J}\mathbf{J}}{\det \mathcal{J}}, \tag{9}$$

$$\rho' = \frac{\rho}{\det \mathcal{J}}, \tag{10}$$

where $\mathcal{J}^T$ is the transpose.

Note that, even if we start out with isotropic materials (scalar ε and μ), after a coordinate transformation we in general obtain *an*isotropic materials (tensors $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\mu}'$).

For example, if $\mathbf{x}' = s\mathbf{x}$ for some scale factor $s \neq 0$, then $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}/s$ and $\boldsymbol{\mu}' = \boldsymbol{\mu}/s$, which is precisely the material scaling required to keep e.g. the eigenfrequencies fixed under a rescaling of a structure. Note also that if $s = -1$, i.e. a coordinate inversion, then we set $\mathbf{E}' = -\mathbf{E}$, $\mathbf{H}' = -\mathbf{H}$, $\boldsymbol{\varepsilon}' = -\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}' = -\boldsymbol{\mu}$, and the system switches "handedness" (flipping the sign of the refractive index). [A more common alternative choice in that case would be to set $\mathbf{H}' = \mathbf{H}$, transforming $\mathbf{H}$ as a pseudovector [6], while keeping $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ unchanged. This corresponds to sprinkling a few factors of $\operatorname{sign}(\det \mathcal{J})$ in the above equations, which we are free to do as long as the sign is constant.]

Proof

We will proceed in index notation, employing the Einstein convention whereby repeated indices are summed over. Eq. (1) is now expressed:

$$\partial_a H_b \epsilon_{abc} = \varepsilon_{cd} \frac{\partial E_d}{\partial t} + J_c \tag{11}$$

where ϵ_{abc} is the usual Levi-Civita permutation tensor and $\partial_a = \partial / \partial x_a$. Under a coordinate change $\mathbf{x} \mapsto \mathbf{x}'$, if we let $\mathcal{J}_{ab} = \frac{\partial x'_a}{\partial x_b}$ be the (non-singular) Jacobian matrix associated with the coordinate transform (which may be a function of $\mathbf{x}$), we have

$$\partial_a = \mathcal{J}_{ba} \partial'_b. \tag{12}$$

Furthermore, as in eqs. (5–6), let

$$E_a = \mathcal{J}_{ba} E'_b, \tag{13}$$
$$H_a = \mathcal{J}_{ba} H'_b. \tag{14}$$

Hence, eq. (11) becomes

$$\mathcal{J}_{ia} \partial'_i \mathcal{J}_{jb} H'_j \epsilon_{abc} = \varepsilon_{cd} \mathcal{J}_{ld} \frac{\partial E'_l}{\partial t} + J_c. \tag{15}$$

Here, the $\mathcal{J}_{ia} \partial'_i = \partial_a$ derivative falls on both the $\mathcal{J}_{jb}$ and H'_j terms, but we can eliminate the former thanks to the ϵ_{abc}: $\partial_a \mathcal{J}_{jb} \epsilon_{abc} = 0$ because $\partial_a \mathcal{J}_{jb} = \partial_b \mathcal{J}_{ja}$. Then, again multiplying both sides by the Jacobian $\mathcal{J}_{kc}$, we obtain

$$\mathcal{J}_{kc} \mathcal{J}_{jb} \mathcal{J}_{ia} \partial'_i H'_j \epsilon_{abc} = \mathcal{J}_{kc} \varepsilon_{cd} \mathcal{J}_{ld} \frac{\partial E'_l}{\partial t} + \mathcal{J}_{kc} J_c \tag{16}$$

Noting that $\mathcal{J}_{ia} \mathcal{J}_{jb} \mathcal{J}_{kc} \epsilon_{abc} = \epsilon_{ijk} \det \boldsymbol{\mathcal{J}}$ by definition of the determinant, we finally have

$$\partial'_i H'_j \epsilon_{ijk} = \frac{1}{\det \boldsymbol{\mathcal{J}}} \mathcal{J}_{kc} \varepsilon_{cd} \mathcal{J}_{ld} \frac{\partial E'_l}{\partial t} + \frac{\mathcal{J}_{kc} J_c}{\det \boldsymbol{\mathcal{J}}} \tag{17}$$

or, back in vector notation,

$$\nabla' \times \mathbf{H}' = \frac{\boldsymbol{\mathcal{J}} \boldsymbol{\varepsilon} \boldsymbol{\mathcal{J}}^T}{\det \boldsymbol{\mathcal{J}}} \frac{\partial \mathbf{E}'}{\partial t} + \mathbf{J}', \tag{18}$$

where $\mathbf{J}' = \boldsymbol{\mathcal{J}} \mathbf{J} / \det \boldsymbol{\mathcal{J}}$ according to (9). Thus, we see that we can interpret Ampere's Law in arbitrary coordinates as the usual equation in Euclidean coordinates, as long as we replace the materials etc. by eqs. (5–7). By an identical argument, we obtain

$$\nabla' \times \mathbf{E}' = -\frac{\boldsymbol{\mathcal{J}} \boldsymbol{\mu} \boldsymbol{\mathcal{J}}^T}{\det \boldsymbol{\mathcal{J}}} \frac{\partial \mathbf{H}'}{\partial t}, \tag{19}$$

which yields the transformation (8) for $\boldsymbol{\mu}$.

The transformation of the remaining divergence equations into equivalent forms in the new coordinates is also straightforward. Gauss' Law, eq. (3), becomes

$$\begin{aligned}\rho = \partial_a \varepsilon_{ab} E_b &= \mathcal{J}_{ia} \partial'_i \varepsilon_{ab} \mathcal{J}_{jb} E'_j \\ &= \mathcal{J}_{ia} \partial'_i (\det \boldsymbol{\mathcal{J}}) \mathcal{J}^{-1}_{ak} \varepsilon'_{kj} E'_j \\ &= (\det \boldsymbol{\mathcal{J}}) \partial'_i \varepsilon'_{ij} E'_j + (\partial_a \mathcal{J}^{-1}_{ak} \det \boldsymbol{\mathcal{J}}) \varepsilon'_{kj} E'_j \\ &= (\det \boldsymbol{\mathcal{J}}) \partial'_i \varepsilon'_{ij} E'_j,\end{aligned} \tag{20}$$

which gives $\nabla' \cdot (\boldsymbol{\varepsilon}' \mathbf{E}') = \rho'$ for $\rho' = \rho / \det \boldsymbol{\mathcal{J}}$, corresponding to eq. (10). Similarly for eq. (4). Here, we have used the fact that

$$\partial_a \mathcal{J}^{-1}_{ak} \det \boldsymbol{\mathcal{J}} = \partial_a \epsilon_{anm} \epsilon_{kij} \mathcal{J}_{in} \mathcal{J}_{jm} / 2 = 0, \tag{21}$$

from the cofactor formula for the matrix inverse, and recalling that $\partial_a \mathcal{J}_{jb} \epsilon_{abc} = 0$ from above. In particular, note that $\rho = 0 \iff \rho' = 0$ and $\mathbf{J} = 0 \iff \mathbf{J}' = 0$, so a non-singular coordinate transformation preserves the absence (or presence) of sources.

References

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