# 18.369 Problem Set 3 Solutions

## Problem 1: (10+15 points)

In both parts of this problem, we need to prove that the Rayleigh quotient satisfies $\langle H, \hat{\Theta}_{\mathbf{k}} H \rangle / \langle H, H \rangle < k^2$ for some trial function $H$, or equivalently that

$$\int_0^a \int_{-\infty}^{\infty} (1-\Delta) \left| (\nabla + i\mathbf{k}) \times \mathbf{H_k} \right|^2 dx\,dy - k^2 \int_0^a \int_{-\infty}^{\infty} |\mathbf{H_k}|^2 \, dx\,dy < 0$$

for the trial Bloch envelope $\mathbf{H_k} = \mathbf{H} e^{-ikx}$, $\mathbf{k} = k\hat{\mathbf{x}}$, and $\varepsilon^{-1} = 1 - \Delta$.

(a) We will choose $u(x,y) = e^{-|y|/L}$ for some $L > 0$, exactly as in class—that is, it is the simplest conceivable periodic function of $x$, a constant. Thus, $\int |u|^2 = 2a \int_0^\infty e^{-2y/L} dy = aL$ over the unit cell. In this case, the variational criterion above becomes, exactly as in class except for the factor of $a$:

$$\begin{aligned} \int_0^a \int_{-\infty}^{\infty} (1-\Delta) \left(k^2 + L^{-2}\right) e^{-2|y|/L} dx dy - k^2 aL &< 0 \\ &= \frac{a}{L} - \int_0^a \int_{-\infty}^{\infty} \Delta \cdot \left(k^2 + L^{-2}\right) e^{-2|y|/L} dx\,dy, \end{aligned}$$

which becomes negative in the limit $L \to \infty$ thanks to our assumption that $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)\, dx\,dy > 0$. Note that the fact that $\int |\Delta| < \infty$ ensures that we can interchange the limits and integration, via the dominated convergence theorem discussed in class.

(b) Let us guess that we can choose $u(y)$ and $v(y)$ to be functions of $y$ only (i.e., again the trivial constant-function periodicity in $x$). The fact that $\nabla \cdot \mathbf{H} = 0$ implies that $(\nabla + i\mathbf{k}) \cdot [u(y)\hat{\mathbf{x}} + v(y)\hat{\mathbf{y}}] = 0 = iku + v'$, and therefore $u = iv'/k$. Therefore, it is convenient to choose $v(y)$ to be a smooth function so that $u$ is differentiable. Let us choose

$$v(y) = e^{-y^2/2L^2}$$

in which case $u(y) = -\frac{iy}{kL^2} e^{-y^2/2L^2}$. Recall the Gaussian integrals $\int_{-\infty}^{\infty} e^{-y^2/L^2} dy = L\sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2/L^2} dy = L^3\sqrt{\pi}/2$. So, $\int |\mathbf{H}|^2 = a \int |u|^2 + |v|^2 = aL\sqrt{\pi}[1 + \frac{1}{k^2L^2}]$. Also, $(\nabla + i\mathbf{k}) \times [u(y)\hat{\mathbf{x}} + v(y)\hat{\mathbf{y}}] = (ikv - u')\hat{\mathbf{z}}$. So,

$$|\nabla \times \mathbf{H}|^2 = |(\nabla + i\mathbf{k}) \times \mathbf{H_k}|^2 = |u'|^2 + k^2|v|^2 = k^2 \left[1 + \frac{1}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)\right] e^{-y^2/L^2}.$$

Then, if we look at our variational criterion, we have two terms: $\int |\nabla \times \mathbf{H}|^2$ and $-\int \Delta \cdot |\nabla \times \mathbf{H}|^2$. Again, we can swap limits with integration in the latter by the dominated convergence theorem. Combining the former with the $-k^2 \int |\mathbf{H}|^2$ term in the variational criterion, we get:

$$\begin{aligned} \int |\nabla \times \mathbf{H}|^2 - k^2 \int |\mathbf{H}|^2 &= a \int_{-\infty}^{\infty} k^2 \left[1 + \frac{1}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right)\right] e^{-y^2/L^2} dy - k^2 aL\sqrt{\pi} \left[1 + \frac{1}{k^2L^2}\right] \\ &= a \int_{-\infty}^{\infty} \frac{k^2}{k^4L^4}\left(1 - \frac{y^2}{L^2}\right) e^{-y^2/L^2} dy - \frac{k^2 aL\sqrt{\pi}}{k^2L^2} \\ &= \frac{a}{k^2L^4} L\sqrt{\pi}\left(1 - \frac{L^2}{2L^2}\right) - \frac{a\sqrt{\pi}}{L}, \end{aligned}$$

which goes to zero as $L \to \infty$. Thus:

$$\int (1-\Delta) \left|(\nabla + i\mathbf{k}) \times \mathbf{H_k}\right|^2 - k^2 \int |\mathbf{H_k}|^2 \to -k^2 \int_0^a \int_{-\infty}^{\infty} \Delta(x,y)\, dx\,dy < 0.$$

as $L \to \infty$. Q.E.D.

## Problem 2: (5+15 points)

(a) Maxwell's equations are (in terms of $\mathbf{H}$) given by the eigen-equation $\nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} = \frac{\omega^2}{c^2} \mathbf{H}$. Suppose that we replace $\varepsilon$ by $\alpha\varepsilon$ where $\alpha$ is some constant. By inspection, one obtains the *same* eigensolution $\mathbf{H}$ with $\omega$ replaced by $\omega/\sqrt{\alpha}$ (we just divided both sides by $\alpha$). Thus, scaling epsilon everywhere by a constant just trivially scales the eigenvalues. [We could have alternatively rescaled the geometry and fields: $\varepsilon(\mathbf{x}) \to \varepsilon(\mathbf{x}\sqrt{\alpha})$ and $\mathbf{H}(\mathbf{x}) \to \mathbf{H}(\mathbf{x}\sqrt{\alpha})$.] Therefore, we can set $\varepsilon_{lo} = 1$ (that is, $\alpha = 1/\varepsilon_{lo}$), without loss of generality.

(b) Since we are looking for "TM" solutions $E_z(x,y) = e^{ikx} E_k(y)$, i.e. with $\mathbf{E}$ in the $z$ direction, then we already saw from the last problem set that the eigen-equation simplifies to $-\nabla^2 E_z = \frac{\omega^2}{c^2} \varepsilon E_z$, and when we plug in the $e^{ikx}$ form we get:

$$-\frac{d^2}{dy^2} E_y = (\omega^2 \varepsilon - k^2) E_y$$

(where I have chosen $c = 1$ units for simplicity).

(i) In any region where $\varepsilon$ is constant, the above equation is solved simply by sines and cosines if $\omega^2\varepsilon - k^2 > 0$ and by exponentials otherwise. Since we have a $y = 0$ mirror plane, the solutions can be chosen either even or odd, and therefore in the $|y| < h/2$ region we have solutions $E_k = A\cos(k_\perp y)$ or $A\sin(k_\perp y)$, where

$$k_\perp = \sqrt{\omega^2 \varepsilon_{hi} - k^2}.$$

If $k_\perp$ is imaginary, these become cosh and sinh solutions, but we will see below that this won't happen. In the $|y| > h/2$ region, since we are looking for solutions below the light line ($\omega^2 \varepsilon_{lo} < k^2$), we must have exponentials...and requiring the solutions to be finite at infinity we must have $E_k = Be^{-\kappa y}$ for $y > h/2$ and $\pm Be^{\kappa y}$ for $y < -h/2$ (with $\pm$ depending on whether the state is even or odd, where:

$$\kappa = \sqrt{k^2 - \omega^2 \varepsilon_{lo}} = \sqrt{k^2(1-f) - k_\perp^2 f},$$

where we define $f = \varepsilon_{lo}/\varepsilon_{hi} < 1$ (the dielectric contrast), and we have used the definition of $k_\perp$ from above.

(ii) Let's consider first the *even* solutions (cosine). Continuity of $E_k$ implies that $A\cos(k_\perp h/2) = Be^{-\kappa h/2}$, and continuity of $E_k' \sim H_x$ implies that $-k_\perp A \sin(k_\perp h/2) = -\kappa B e^{-\kappa h/2}$. Dividing these two equations, we find:

$$\tan(k_\perp h/2) = \frac{\kappa}{k_\perp} = \frac{\sqrt{k^2(1-f) - k_\perp^2 f}}{k_\perp}.$$

Similarly, for the *odd* solutions (sine), we obtain:

$$\cot(k_\perp h/2) = -\frac{\sqrt{k^2(1-f) - k_\perp^2 f}}{k_\perp}.$$

These are transcendental equations for $k_\perp$. We plot the left and right hand sides of these two equations in figure 1, where the intersections of the curves give the guided-mode solutions.

What about imaginary $k_\perp$ solutions? In this case, the left hand side (tan or cot) would be purely imaginary, while the right hand side would also be purely imaginary, so it seems like there might be some such solutions. Consider the even mode (tan) equation. The tangent of an imaginary $k_\perp$ is always imaginary with the *same* sign as the imaginary part of $k_\perp$, whereas the right hand side will be imaginary with the *opposite* sign ($1/i = -i$)—because of that, the two curves will *never* intersect for imaginary $k_\perp$ and there will be no solution. Conversely for the odd-mode

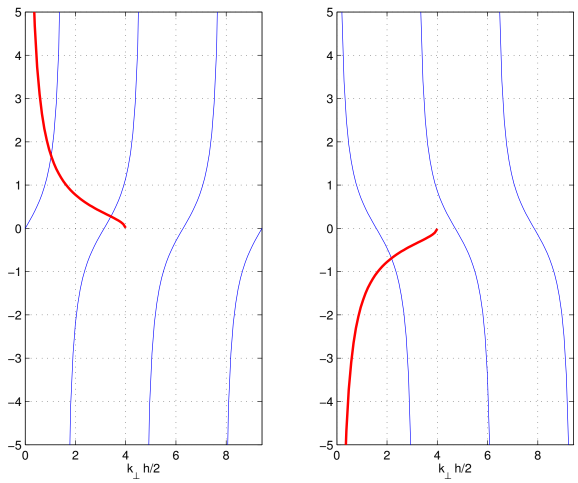

Figure 1: Plot of the two transcendental equations for even modes (left plot) and odd modes (right plot) as a function of $k_\perp h/2$. The thick lines show the right hand sides, while the thin lines show the left hand sides (tan or cot) of the equations, and the intersections correspond to guided-mode solutions. This plot is for the particular case of $f = 0.1$ and $kh/2 = 2$.

case. So, there are no imaginary $k_\perp$ solutions, as promised—this means that the guided modes must always be *above* the light line for $\varepsilon_{hi}$, which makes physical sense (they must correspond to *propagating* modes in the $\varepsilon_{hi}$ region and *evanescent* modes in the $\varepsilon_{lo}$ regions).

(iii) We can see immediately that the right-hand side of the transcendental equations is a real number only when $k_\perp \leq |k|\sqrt{\frac{1}{f} - 1} = k_\perp^{max}$. Furthermore, we will clearly have an intersection for *every* branch of the tangent/cotangent curve that passes through zero *before* $k_\perp^{max}$. The tangent curves pass through zero whenever $k_\perp h/2$ is an integer multiple of $\pi$, and the cotangent curves pass through zero when $k_\perp h/2 + \pi/2$ is an integer multiple of $\pi$. Therefore, the number of even modes is simply the number of zero crossings before $k_\perp^{max}$, namely:

$$\text{\# even modes} = \left\lfloor \frac{|k|h\sqrt{\frac{1}{f} - 1}}{2\pi} \right\rfloor + 1,$$

where the $+1$ is for the first branch of the tangent (which has a zero crossing at $k_\perp = 0$ and therefore *always* intersects the right-hand-side at least once). Here, by $\lfloor x \rfloor$ we mean the greatest[1] integer $\leq x$. Similarly, the number of odd modes is also given by the number of zero crossings:

$$\text{\# odd modes} = \left\lfloor \frac{|k|h\sqrt{\frac{1}{f} - 1} + \pi}{2\pi} \right\rfloor,$$

where in this case we see that we will not have *any* odd guided modes for $|k|h\sqrt{\frac{1}{f} - 1} < \pi$. Therefore, as $k \to 0$ we get exactly one (even) guided mode.

Just for fun, let's look at the TE polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). For the $H_z = H_k e^{ikx}$ polarization, we have very similar equations except that the boundary conditions are that $H_k$ is continuous and $H_k'/\varepsilon$ is continuous

[1] There is some ambiguity about whether to define the mode as guided when the argument of $\lfloor x \rfloor$ here is exactly an integer, because that corresponds to the case where the mode is exactly on the light line and hence has $\kappa = 0$. If we don't call that a guided mode, then we have to modify our formula by one in that case, but since this situation has measure zero in the parameter space, the question has no practical significance.

(since $H_k' \sim D_x = D_\parallel$). Thus, for example for the $\cos(k_\perp y)$ mode (the *odd* mode, since $\mathbf{H}$ is a pseudovector), we have $-k_\perp A \sin(k_\perp h/2)/\varepsilon_{hi} = -\kappa B e^{-\kappa h/2}/\varepsilon_{lo}$. Therefore, both the tan and cot in the transcendental equations get multipled by $f = \varepsilon_{lo}/\varepsilon_{hi}$. What effect does this have on the solutions? Multiplying by $f < 1$ *decreases* the tangent curves, but does *not* change the locations of their zeros. Therefore, the *number* of modes at a given $k$ is *unaffected*. However, the intersection point is clearly pulled towards *larger* values of $k_\perp$ when the tan/cot is shrunk, which corresponds to *smaller* values of $\kappa$, the decay rate. Therefore, the modes are *less* strongly confined for the $H_z$ (TE) polarization. (Later in the class, we will see how this generally follows from the boundary conditions and the variational theorem.)

## Problem 3: (10+20 points)

(a) Trivially from the given identity,

$$\mathbf{J} = \frac{a}{2\pi}\int_0^{2\pi/a}\left[\sum_{n=-\infty}^{\infty}\delta(x-na)\delta(y)e^{ikna-i\omega t}\hat{z}\right]dk,$$

where the term $[\cdots]$ is Bloch-periodic. Because irrep is conserved, from class and homework, the resulting steady-state/time-harmonic $\mathbf{E}$ field from each Bloch-periodic term is also Bloch-periodic (from periodicity) and TM-polarized (from $z \leftrightarrow -z$ mirror symmetry), i.e. the field of the current $[\cdots]$ is:

$$\mathbf{E}_k(x,y,t) = E_k(x,y)e^{ikx-i\omega t}\hat{z},$$

where $E_k(x+a,y) = E_k(x,y)$. By linearity, we can simply add up the solutions $\mathbf{E}_k$ from the integrand of $\mathbf{J}$ to get the total field by superposition:

$$\mathbf{E} = \frac{a}{2\pi}\int_0^{2\pi/a}\mathbf{E}_k dk = \hat{z}\frac{a}{2\pi}\int_0^{2\pi/a}E_k(x,y)e^{ikx-i\omega t}dk.$$

The field $\mathbf{E}_k$ satisfies $(\nabla\times\nabla\times - \omega^2\varepsilon)\mathbf{E}_k = i\omega\mathbf{J}$, and since both $\mathbf{E}_k$ and $\mathbf{J}$ are Bloch-periodic we can trivially reduce the domain to the unit cell $(x,y) \in [0,a]\times(-\infty,\infty)$ with Bloch-periodic boundary conditions.

(b) To get the total power, we also need

$$\mathbf{H} = \frac{1}{i\omega}\nabla\times\mathbf{E} = \frac{a}{2\pi}\int_0^{2\pi/a}\mathbf{H}_k(x,y)e^{ikx-i\omega t}dk,$$

where $\mathbf{H}_k = \nabla_k \times (E_k\hat{z})$ is a periodic function. Hence, $\mathbf{H}$, like $\mathbf{E}$, is a superposition of Bloch-periodic functions. Because partners of different irreps are orthogonal, the $\int dx$ of $\hat{y}\cdot(E_k\hat{z}e^{ikx})^* \times (\mathbf{H}_{k'}e^{ik'x})$ must necessarily be zero unless $k = k'$, hence the total power $P$ will be a superposition of terms $P_k$ that are integrals of $\hat{y}\cdot(E_k\hat{z}e^{ikx})^*\times(\mathbf{H}_k e^{ikx})$: the Poynting flux of one $k$ at a time.

However, getting the exact form of this superposition, including the normalization, is a bit tricky because of the infinite bounds of the $x$ integral. One approach is to try and use the Fourier identity $\int_{-\infty}^{\infty} e^{i(k-k')x}dx = 2\pi\delta(k-k')$. However, applying this requires a little bit of care because the envelopes $E_k$ and $\mathbf{H}_k$ are periodic in $x$, not constants. In order to perform the $x$ integral and apply the Fourier identity, we can expand the periodic function $\hat{y}\cdot(E_k\hat{z})^*\times(\mathbf{H}_{k'})$ in a Fourier series:

$$p(k,k',x) = \hat{y}\cdot E_k(x)^*\hat{z}\times\mathbf{H}_{k'}(x) = \sum_{n=-\infty}^{\infty}\hat{p}_n(k,k')e^{i\frac{2\pi n}{a}x}$$

(where we have evaluated everything at $y = y_0$). Plugging this in, we obtain (being somewhat casual about interchanging the order of integrals/sums):

$$
\begin{aligned}
P &= \frac{1}{2}\int_{-\infty}^{\infty} \hat{y}\cdot\Re\left[\mathbf{E}^*(x,y_0)\times\mathbf{H}(x,y_0)\right]dx \\
&= \frac{1}{2}\hat{y}\cdot\Re\frac{a}{2\pi}\int_0^{2\pi/a} dk\frac{a}{2\pi}\int_0^{2\pi/a} dk'\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}\hat{p}_n(k,k')e^{i(k'-k+\frac{2\pi n}{a})x}\right]dx \\
&= \frac{1}{2}\hat{y}\cdot\Re\frac{a}{2\pi}\int_0^{2\pi/a} dk\frac{a}{2\pi}\int_0^{2\pi/a} dk'\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}\hat{p}_n(k,k')2\pi\delta(k'-k+\frac{2\pi n}{a})\right]dx.
\end{aligned}
$$

Now, the delta function is only nonzero for $n=0$ in the sum, since $|k'-k|<2\pi/a$ for $k,k'\in[0,2\pi/a)$. Thus, we obtain:

$$
P = \frac{a^2}{2\pi}\frac{1}{2}\hat{y}\cdot\Re\int_0^{2\pi/a}\hat{p}_0(k,k)dk,
$$

but since $\hat{p}_0(k,k') = \frac{1}{a}\int_0^a \hat{y}\cdot E_k(x)^*\hat{z}\times\mathbf{H}_k(x)dx$ by the usual Fourier-series formula, we find

$$
\boxed{P = \frac{a}{2\pi}\int_0^{2\pi/a} P_k dk}
$$

where

$$
\boxed{P_k = \frac{1}{2}\hat{y}\cdot\Re\int_0^a\left[\hat{z}E_k(x,y_0)\right]^*\times\left[\mathbf{H}_k(x,y_0)\right]dx}
$$

is the Poynting flux in a single unit cell from the Bloch solution for a single $k$.

Alternatively, another approach is to adopt a trick that we will use to analyze the per-period LDOS in class: instead of an infinite system in the $x$ direction, we consider a supercell of $M$ periods, with periodic boundaries $x\leftrightarrow x+Ma$, and take the $M\to\infty$ limit in the end. As in the LDOS notes, for such a supercell we get a subset of the Bloch solutions, only $k_m = \frac{2\pi}{Ma}m$ for integers $m=0,\ldots,P-1$, or equivalently one can easily show:

$$
\mathbf{J} = \frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{n=0}^{M-1}\delta(x-na)\delta(y)e^{ik_m na - i\omega t}\hat{z}\right],
$$

$$
\mathbf{E} = \hat{z}\frac{1}{M}\sum_{m=0}^{M-1}E_{k_m}(x,y)e^{ik_m x - i\omega t}
$$

for the *same* $E_k$ solutions as above. The Poynting flux is then

$$
\begin{aligned}
P &= \frac{1}{2}\int_0^{Ma}\hat{y}\cdot\Re\left[\mathbf{E}^*(x,y_0)\times\mathbf{H}(x,y_0)\right]dx \\
&= \frac{1}{2}\hat{y}\cdot\Re\int_0^{Ma}\left[\hat{z}\frac{1}{M}\sum_{m=0}^{M-1}E_{k_m}(x,y_0)e^{ik_m x-i\omega t}\right]^*\times\left[\frac{1}{M}\sum_{m'=0}^{M-1}\mathbf{H}_{k_{m'}}(x,y_0)e^{ik_{m'}x-i\omega t}\right]dx.
\end{aligned}
$$

As above, the $m\neq m'$ cross terms must integrate to zero (since they are partner functions of different irreps of the symmetry group: translations by $na$ for $n=0,\ldots,M-1$). What remains is

$$
\begin{aligned}
P &= \frac{1}{2}\hat{y}\cdot\Re\frac{1}{M^2}\sum_{m=0}^{M-1}\int_0^{Ma}\left[\hat{z}E_{k_m}(x,y_0)\right]^*\times\left[\mathbf{H}_{k_m}(x,y_0)\right]dx \\
&= \frac{1}{2}\hat{y}\cdot\Re\frac{1}{M}\sum_{m=0}^{M-1}\int_0^{a}\left[\hat{z}E_{k_m}(x,y_0)\right]^*\times\left[\mathbf{H}_{k_m}(x,y_0)\right]dx \\
&= \frac{1}{M}\sum_{m=0}^{M-1}P_{k_m},
\end{aligned}
$$

where we have used the periodicity of $E_k$ and $\mathbf{H}_k$ and defined $P_k$ as above. Finally, by multiplying and dividing by $\Delta k = \frac{2\pi}{Ma}$ as in class, we can take the $M \to \infty$ limit to recover the integral $P = \frac{a}{2\pi}\int_0^{2\pi/a} P_k dk$ as above.