18.369 Problem Set 1 Solutions

Problem 1: Adjoints and operators (5+10+5 points)

(a) If \( h \) is a column-vector and we let \( h^\dagger \) be its conjugate transpose, then \( h^\dagger \) is a row vector and \( h^\dagger h' = \sum_m h^*_m h'_m = \langle h, h' \rangle \) by the usual row-times-column multiplication rule. If \( O \) is a matrix then \( Oh = \sum_n O_{mn} h'_n \) by the usual matrix-vector product. Then the dot product of \( h \) with \( Oh' \) is given by \( \sum_m h^*_m (\sum_n O_{mn} h'_n) = \sum_n (\sum_m O_{mn} h_m^*) h'_n \), which is the same thing as the dot product of \( O^\dagger h \) with \( h' \) where \( O^\dagger \) is the conjugate transpose of \( O \). Thus, interpreting \( \dagger \) as the conjugate transpose in this finite-dimensional case is consistent with the abstract definitions given in class.

(b) If \( \hat{O} \) is unitary and we send \( u \rightarrow \hat{O}u \) and \( v \rightarrow \hat{O}v \), then \( \langle u, v \rangle \rightarrow \langle u, \hat{O}^\dagger \hat{O}v \rangle = \langle u, v \rangle \), and thus inner products are preserved. Consider now two eigensolutions \( \hat{O}u_1 = \lambda_1 u_1 \) and \( \hat{O}u_2 = \lambda_2 u_2. \) Then \( \langle u_1, \hat{O}^\dagger \hat{O}u_2 \rangle = \langle u_1, u_2 \rangle \) by the unitarity of \( \hat{O} \) and \( \langle u_1, \hat{O}^\dagger \hat{O}u_2 \rangle = \langle \hat{O}u_1, \hat{O}u_2 \rangle = \lambda_1^* \lambda_2 \langle u_1, u_2 \rangle \) by the eigenvector property (where we let \( \hat{O}^\dagger \) act to the left, and conjugate the eigenvalue when we factor it out, as in class). Combining these two expressions, we have \( (\lambda_1^* \lambda_2 - 1) \langle u_1, u_2 \rangle = 0 \). There are three cases, just like for Hermitian operators. If \( u_1 = u_2 \), then we must have \( \lambda_1^* \lambda_2 = 1 = |\lambda_1|^2 \), and thus the eigenvalues have unit magnitude. This also implies that \( \lambda_1^* = 1/\lambda_1 \). If \( \lambda_1 \neq \lambda_2 \), then \( \lambda_1^* \lambda_2 - 1 = (\lambda_2/\lambda_1 - 1) \neq 0 \), and therefore \( \langle u_1, u_2 \rangle = 0 \) and the eigenvectors are orthogonal. If \( \lambda_1 = \lambda_2 \) but have linearly independent \( u_1 \neq u_2 \) (degenerate eigenvectors, i.e. geometric multiplicity \( > 1 \)), then we can form orthogonal linear combinations (e.g. via Gram–Schmidt).

(c) Take two vectors \( u \) and \( v \), and consider their inner product. Then \( \langle u, \hat{O}^{-1} \hat{O}v \rangle = \langle u, v \rangle. \) By definition of the adjoint, however, if we move first \( \hat{O}^{-1} \) and then \( \hat{O} \) to act to the left, then we get \( \langle u, v \rangle = \langle \hat{O}^\dagger \hat{O}^{-1} \rangle u, v \). For this to be true for all \( u \) and \( v \), we must have \( \hat{O}^\dagger \hat{O}^{-1} \) and \( \hat{O}^{-1} \) and thus \( \hat{O}^\dagger = (\hat{O}^{-1})^{-1}. \) Q.E.D.

Problem 2: Maxwell eigenproblems (5+5+5+5+5 points)

(a) To eliminate \( H \), we start with Faraday’s law \( \nabla \times E = i \omega/c H \) and take the curl of both sides. We obtain:

\[
\nabla \times \nabla \times E = \frac{\omega^2}{c^2} \varepsilon E.
\]

If we divide both sides by \( \varepsilon \), we get the form of a linear eigenproblem but the operator \( \frac{1}{i} \nabla \times \nabla \times \) is not Hermitian under the usual inner product \( \langle E_1, E_2 \rangle = \int E_1^\* \cdot E_2 \) —integrating by parts as in class, assuming boundary conditions such that the boundary terms vanish, we find that its adjoint is \( \nabla \times \nabla \times \frac{1}{i} \), which is not the same operator unless the \( \frac{1}{i} \) commutes with the curls, which only happens if \( \varepsilon \) is a constant. However, if we
leave it in the form above we have a generalized Hermitian problem with $A = \nabla \times \nabla \times$ and $B = \varepsilon$. $A$ is Hermitian for the same reason that $\Theta$ was (it is $\Theta$ for $\varepsilon = 1$), and $B$ is Hermitian as long as $\varepsilon$ is real (so that $H_1^* \cdot \varepsilon H_2 = (\varepsilon H_1)^* \cdot H_2$).

(b) The proof follows the same lines as in class. Consider two eigensolutions $u_1$ and $u_2$ (where $Au = \lambda Bu$, and $u \neq 0$), and take $\langle u_2, Au_1 \rangle$. Since $A$ is Hermitian, we can operate it to the left or to the right in the inner product, and get $\lambda_1^* \langle u_2, Bu_1 \rangle = \lambda_1 \langle u_2, Bu_1 \rangle$, or $\langle u_2, Bu_1 \rangle = 0$. There are three cases. First, if $u_1 = u_2$ then we must have $\lambda_1 = \lambda_1^*$ (real eigenvalues), since $\langle u_1, Bu_1 \rangle > 0$ by definition if $B$ is positive definite. Second, if $\lambda_1 \neq \lambda_2$ then we must have $\langle u_2, Bu_1 \rangle = 0$, which is our modified orthogonality condition. Finally, if $\lambda_1 = \lambda_2$ but $u_1 \neq u_2$, then we can form a linear combination that is orthogonal (since any linear combination still is an eigenvector); e.g.

$$u_2 \rightarrow u_2 - u_1 \frac{\langle u_2, Bu_1 \rangle}{\langle u_1, Bu_1 \rangle},$$

where we have again relied on the fact that $B$ is positive definite (so that we can divide by $\langle u_1, Bu_1 \rangle$). This is certainly true for $B = \varepsilon$, since $\langle E, B \varepsilon E \rangle = \int \varepsilon |E|^2 > 0$ for all $E \neq 0$ (almost everywhere) as long as $\varepsilon > 0$ as we assumed in class.

(c) First, let us verify that $\langle E, B \varepsilon E \rangle = \langle E, B \varepsilon E \rangle$ is indeed an inner product. Because $B$ is self-adjoint, we have $\langle E', E \rangle_B = \langle E', B \varepsilon E \rangle = \langle B \varepsilon E', E \rangle = \langle E, B \varepsilon E' \rangle^* = \langle E, E' \rangle_B$. Bilinearity follows from bilinearity of $\langle \cdot, \cdot \rangle$ and linearity of $B$. Positivity $\langle E, E \rangle_B = \langle E, B \varepsilon E \rangle > 0$ except for $E = 0$ (almost everywhere) follows from positive-definiteness of $B$. All good!

Now, Hermiticity of $B^{-1} A$ follows almost trivially from Hermiticity of $A$ and $B$: $(E, B^{-1} A E')_B = (E, B B^{-1} A E') = \langle A E, E' \rangle = \langle A E, B^{-1} B E' \rangle = \langle B^{-1} A E, B E' \rangle = \langle B^{-1} A E, E' \rangle_B$, where we have used the fact, from problem 1, that Hermiticity of $B$ implies Hermiticity of $B^{-1}$. Q.E.D.

(d) If $\mu \neq 1$ then we have $B = \mu H \neq H$, and when we eliminate $E$ or $H$ from Maxwell’s equations we get:

$$\nabla \times \frac{1}{\varepsilon} \nabla \times H = \frac{\omega^2}{c^2} \mu H$$

$$\nabla \times \frac{1}{\mu} \nabla \times E = \frac{\omega^2}{c^2} \varepsilon E$$

with the constraints $\nabla \cdot \varepsilon E = 0$ and $\nabla \cdot \mu H = 0$. These are both generalized Hermitian eigenproblems (since $\mu$ and $\nabla \times \frac{1}{\mu} \nabla \times$ are both Hermitian operators for the same reason $\varepsilon$ and $\nabla \times \frac{1}{\varepsilon} \nabla \times$ were). Thus, the eigenvalues are real and the eigenstates are orthogonal through $\mu$ and $\varepsilon$, respectively,
as proved above. To prove that $\omega$ is real, we consider an eigenfunction $H$. Then $\langle H, \hat{\Theta} H \rangle = \frac{\omega^2}{\varepsilon} \langle H, \mu H \rangle$ and we must have $\omega^2 \geq 0$ since $\hat{\Theta}$ is positive semi-definite (from class) and $\mu$ is positive definite (for the same reason $\varepsilon$ was, above). The $E$ eigenproblem has real $\omega$ for the same reason (except that $\mu$ and $\varepsilon$ are swapped).

Alternatively, as in part (c), we can write them as ordinary Hermitian eigenproblems with a modified inner product, e.g. $\frac{1}{\varepsilon} \nabla \times \frac{1}{\mu} \nabla \times E = \frac{\omega^2}{\varepsilon} E$, where $\frac{1}{\varepsilon} \nabla \times \frac{1}{\mu} \nabla \times$ is Hermitian and positive semidefinite under the inner product $\langle E, E' \rangle_B = \int E^* \cdot \varepsilon E'/\mu$ inner product as above. The results then follow.

(c) Consider the $H$ eigenproblem. (To even get this linear eigenproblem, we must immediately require $\varepsilon$ to be an invertible matrix, and of course require $\varepsilon$ and $\mu$ to be independent of $\omega$ or the field strength.) For the right-hand operator $\mu$ to be Hermitian, we require $\int H_1^* \cdot \mu H_2 = \int (\mu H_1)^* \cdot H_2$ for all $H_1$ and $H_2$, which implies that $H_1^* \cdot \mu H_2 = (\mu H_1)^* \cdot H_2$. Thus, we require the $3 \times 3$ $\mu(x)$ matrix to be itself Hermitian at every $x$ (that is, equal to its conjugate transpose, from problem 1). (Technically, these requirements hold “almost everywhere” rather than at every point, but as usual I will gloss over this distinction.) Similarly, for $\hat{\Theta}$ to be Hermitian we require $\int F_1^* \cdot \varepsilon^{-1} F_2 = \int (\varepsilon^{-1} F_1)^* \cdot F_2$ where $F = \nabla \times H$, so that we can move the $\varepsilon^{-1}$ over to the left side of the inner product, and thus $\varepsilon^{-1}(x)$ must be Hermitian at every $x$. From problem 1, this implies that $\varepsilon(x)$ is also Hermitian. Finally, to get real eigenvalues we saw from above that we must have $\mu$ positive definite ($\int H^* \cdot \mu H > 0$ for $H \neq 0$); since this must be true for all $H$ then $\mu(x)$ at each point must be a positive-definite $3 \times 3$ matrix (positive eigenvalues). Similarly, $\hat{\Theta}$ must be positive semi-definite, which implies that $\varepsilon^{-1}(x)$ is positive semi-definite (non-negative eigenvalues), but since it has to be invertible we must have $\varepsilon(x)$ positive definite (zero eigenvalues would make it singular). To sum up, we must have $\varepsilon(x)$ and $\mu(x)$ being positive-definite Hermitian matrices at (almost) every $x$. (The analysis for the $E$ eigenproblem is identical.)

Technically, there are a couple other possibilities. In part (b), we showed that if $\hat{B}$ is positive-definite it leads to real eigenvalues etc. The same properties, however, hold if $\hat{B}$ is negative-definite, and if both $\hat{A}$ and $\hat{B}$ are negative-definite we still get real, positive eigenvalues. Thus, another possibility is for $\varepsilon$ and $\mu$ to be Hermitian negative-definite matrices. (For a scalar $\varepsilon < 0$ and $\mu < 0$, this leads to materials with a negative real index of refraction $n = -\sqrt{\varepsilon\mu}$) Furthermore, $\varepsilon$ and $\mu$ could both be anti-Hermitian instead of Hermitian (i.e., $\varepsilon^\dagger = -\varepsilon$ and $\mu^\dagger = -\mu$), although I’m not aware of any physical examples of this. More generally, for any complex number $z$, if we replace $\varepsilon$ and $\mu$ by $z\varepsilon$ and $\mu/z$, then $\omega$ is unchanged (e.g. making $z = i$ gives anti-Hermitian matrices).