Problem 1: Adjoints and operators

(a) If \( h \) is a column-vector and we let \( h^\dagger \) be its conjugate transpose, then \( h^\dagger \) is a row vector and \( h^\dagger h' = \sum_m h_m^* h_m' = \langle h, h' \rangle \) by the usual row-times-column multiplication rule. If \( \hat{O} \) is a matrix then \( \hat{O} h' = \sum_n O_{mn} h_n' \) by the usual matrix-vector product. Then the dot product of \( h \) with \( \hat{O} h' \) is given by \( \sum_m h_m^* (\sum_n O_{mn} h_n') = \sum_n (\sum_m O_{mn}^* h_m') h_n' \), which is the same thing as the dot product of \( \hat{O}^\dagger h \) with \( h' \) where \( \hat{O}^\dagger \) is the conjugate transpose of \( \hat{O} \). Thus, interpreting \( \dagger \) as the conjugate transpose in this finite-dimensional case is consistent with the abstract definitions given in class.

(b) We simply rely on two required properties of inner products: conjugacy \( \langle u, v \rangle = \langle v, u \rangle^* \), and linearity \( \langle u, \lambda v \rangle = \lambda \langle u, v \rangle \). Then, \( \langle u, \hat{O}^\dagger v \rangle = \langle u, \hat{O} v \rangle^* = \langle v, \hat{O}^\dagger u \rangle = \langle v, \hat{O}^\dagger \hat{O} u \rangle = \langle v, \hat{O} \hat{O}^\dagger u \rangle = \langle \hat{O}^\dagger u, v \rangle \). Thus, inspecting the definition of \( \hat{O}^\dagger \), we see that \( \hat{O}^\dagger = \hat{O}^* \).

(c) If \( \hat{O} \) is unitary and we send \( u \to \hat{O} u \) and \( v \to \hat{O} v \), then \( \langle u, v \rangle \to \langle u, \hat{O}^\dagger \hat{O} v \rangle = \langle u, v \rangle \), and thus inner products are preserved. Consider now two eigenstates \( \hat{O} u_1 = \lambda_1 u_1 \) and \( \hat{O} u_2 = \lambda_2 u_2 \). Then \( \langle u_1, \hat{O} \hat{O}^\dagger u_2 \rangle = \langle u_1, u_2 \rangle \) by the unitarity of \( \hat{O} \) and \( \langle u_1, \hat{O}^\dagger \hat{O} u_2 \rangle = \langle \hat{O}^\dagger u_1, \hat{O} u_2 \rangle = \lambda_2^* \lambda_1 \langle u_1, u_2 \rangle \) by the eigenvector property (where we let \( \hat{O} \) act to the left, and conjugate the eigenvalue as proved above). Combining these two expressions, we have \( (\lambda_1^\dagger \lambda_2 - 1) \langle u_1, u_2 \rangle = 0 \). There are three cases, just like for Hermitian operators. If \( u_1 = u_2 \), then we must have \( \lambda_1^\dagger \lambda_1 = 1 = |\lambda_1|^2 \), and thus the eigenvalues have unit magnitude. This also implies that \( \lambda_1^\dagger = 1/\lambda_1 \). If \( \lambda_1 \neq \lambda_2 \), then \( (\lambda_1^\dagger \lambda_2 - 1) = (\lambda_2/\lambda_1 - 1) \neq 0 \), and therefore \( \langle u_1, u_2 \rangle = 0 \) and the eigenstates are orthogonal. If \( \lambda_1 = \lambda_2 \) but \( u_1 \neq u_2 \) (degenerate eigenstates), then we can form orthogonal linear combinations.

(d) Take two states \( u \) and \( v \), and consider the inner product. Then \( \langle u, \hat{O}^\dagger \hat{O} v \rangle = \langle u, v \rangle \). By definition of the adjoint, however, if we move first \( \hat{O}^{-1} \) and then \( \hat{O} \) to act to the left, then we get \( \langle u, v \rangle = \langle \hat{O}^\dagger (\hat{O}^{-1})^\dagger, v \rangle \). For this to be true for all \( u \) and \( v \), we must have \( \hat{O}^\dagger (\hat{O}^{-1})^\dagger = 1 \) and thus \( (\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1} \). Q.E.D.

Problem 2: Maxwell eigenproblems

(a) To eliminate \( \mathbf{H} \), we start with Faraday’s law \( \nabla \times \mathbf{E} = \frac{i \omega}{c} \mathbf{H} \) and take the curl of both sides. We obtain:

\[
\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \varepsilon \mathbf{E}.
\]

If we divide both sides by \( \varepsilon \), we get the form of a linear eigenproblem but the operator \( \frac{1}{\varepsilon} \nabla \times \nabla \times \) is not Hermitian—integrating by parts, we find that its adjoint is \( \nabla \times \nabla \times \frac{1}{\varepsilon} \), which is not the same operator unless the \( \frac{1}{\varepsilon} \)
commutes with the curls, which only happens if $\epsilon$ is a constant. However, if we leave it in the form above we have a generalized Hermitian problem with $A = \nabla \times \nabla \times$ and $\hat{B} = \epsilon \cdot A$ is Hermitian for the same reason that $\hat{\Theta}$ was (it is $\hat{\Theta}$ for $\epsilon = 1$), and $\hat{B}$ is Hermitian as long as $\epsilon$ is real (so that $H_1^* \cdot H_2 = (\epsilon H_1)^* \cdot H_2$).

(b) The proof follows the same lines as in class. Consider two eigenstates $u_1$ and $u_2$ (where $\hat{A}u = \lambda \hat{B}u$), and take $\langle u_2, \hat{B}u_1 \rangle$. Since $\hat{A}$ is Hermitian, we can operate it to the left or to the right in the inner product, and get $\lambda_2^2 \langle u_2, \hat{B}u_1 \rangle = \lambda_1 \langle u_2, \hat{B}u_1 \rangle$, or $(\lambda_2^2 - \lambda_1) \langle u_2, \hat{B}u_1 \rangle = 0$. There are three cases. First, if $\lambda_1 = \lambda_2$ then we must have $\lambda_1 = \lambda_1^*$ (real eigenvalues), since $\langle u_1, \hat{B}u_1 \rangle > 0$ by definition if $\hat{B}$ is positive definite. Second, if $\lambda_1 \neq \lambda_2$ then we must have $\langle u_2, \hat{B}u_1 \rangle = 0$, which is our modified orthogonality condition. Finally, if $\lambda_1 = \lambda_2$ but $u_1 \neq u_2$, then we can form a linear combination that is orthogonal (since any linear combination still is an eigenvector); e.g.

$$u_2 \rightarrow u_2 - u_1 \frac{\langle u_2, \hat{B}u_1 \rangle}{\langle u_1, \hat{B}u_1 \rangle},$$

where we have again relied on the fact that $\hat{B}$ is positive definite (so that we can divide by $\langle u_1, \hat{B}u_1 \rangle$). This is certainly true for $\hat{B} = \epsilon \cdot A$, since $\langle \epsilon \cdot E, \hat{B} \epsilon \cdot E' \rangle = \int \epsilon |E|^2 > 0$ for all $\epsilon \neq 0$ as long as $\epsilon > 0$ as we assumed in class.

(c) First, let us verify that $\langle \epsilon \cdot E, \epsilon \cdot E' \rangle_B = \langle \epsilon \cdot E, \hat{B} \epsilon \cdot E' \rangle$ is indeed an inner product. Because $\hat{B}$ is self-adjoint, we have $\langle \epsilon \cdot E, \epsilon \cdot E' \rangle_B = \langle \epsilon \cdot E', \hat{B} \epsilon \cdot E \rangle = \langle \epsilon \cdot E, \hat{B} \epsilon \cdot E' \rangle_B$. Bilinearity follows from bilinearity of $\langle \cdot, \cdot \rangle$ and linearity of $\hat{B}$. Positivity $\langle \epsilon \cdot E, \epsilon \cdot E \rangle_B = \langle \epsilon \cdot E, \hat{B} \epsilon \cdot E \rangle > 0$ except for $\epsilon \cdot E = 0$ (almost everywhere) follows from positive-definiteness of $\hat{B}$. All good!

Now, Hermiticity of $\hat{B}^{-1} \hat{A}$ follows almost trivially from Hermiticity of $\hat{A}$ and $\hat{B}$: $\langle \epsilon \cdot E, \hat{B}^{-1} \hat{A} \epsilon \cdot E' \rangle_B = \langle \epsilon \cdot E, \hat{B} \hat{B}^{-1} \hat{A} \epsilon \cdot E' \rangle = \langle \epsilon \cdot E, \hat{A} \epsilon \cdot E' \rangle = \langle \hat{B} \hat{B}^{-1} \hat{A} \epsilon \cdot E', \epsilon \cdot E \rangle_B = \langle \hat{B}^{-1} \hat{A} \epsilon \cdot E', \epsilon \cdot E \rangle_B$, where we have used the fact, from problem 1, that Hermiticity of $\hat{B}$ implies Hermiticity of $\hat{B}^{-1}$. Q.E.D.

(d) If $\mu \neq 1$ then we have $\hat{B} = \mu \hat{H} \neq \hat{H}$, and when we eliminate $\epsilon \cdot E$ or $\epsilon \cdot H$ from Maxwell’s equations we get:

$$\nabla \times \frac{1}{\epsilon} \nabla \times \hat{H} = \frac{\omega^2}{c^2} \mu \hat{H}$$

$$\nabla \times \frac{1}{\mu} \nabla \times \hat{E} = \frac{\omega^2}{c^2} \epsilon \hat{E}$$

with the constraints $\nabla \cdot \epsilon \hat{E} = 0$ and $\nabla \cdot \mu \hat{H} = 0$. These are both generalized Hermitian eigenproblems (since $\mu$ and $\nabla \times \frac{1}{\mu} \nabla \times$ are both Hermitian operators for the same reason $\epsilon$ and $\nabla \times \frac{1}{\epsilon} \nabla \times$ were); we can’t make them
ordinary Hermitian eigenproblems for the same reason as in the E eigen-
problem above, except in the trivial case of $\mu$ or $\varepsilon$ constant. Thus, the
eigenvalues are real and the eigenstates are orthogonal through $\mu$ and $\varepsilon$,
respectively, as proved above. To prove that $\omega$ is real, we consider an
eigenstate $H$. Then $\langle H, \hat{\Theta} H \rangle = \frac{\omega^2}{\mu} \langle H, \mu H \rangle$ and we must have $\omega^2 \geq 0$
since $\hat{\Theta}$ is positive semi-definite (from class) and $\mu$ is positive definite (for
the same reason $\varepsilon$ was, above). The E eigenproblem has real $\omega$ for the
same reason (except that $\mu$ and $\varepsilon$ are swapped).

(e) Consider the H eigenproblem. (To even get this linear eigenproblem, we
must immediately require $\varepsilon$ to be an invertible matrix, and of course re-
quire $\varepsilon$ and $\mu$ to be independent of $\omega$ or the field strength.) For the right-
hand operator $\mu$ to be Hermitian, we require $\int H_1 \cdot \mu H_2 = \int (\mu H_1)^* \cdot H_2$
for all $H_1$ and $H_2$, which implies that $H_1 \cdot \mu H_2 = (\mu H_1)^* \cdot H_2$. Thus,
we require the $3 \times 3 \mu(x)$ matrix to be itself Hermitian at every $x$ (that
is, equal to its conjugate transpose, from problem 1). (Technically, these
requirements hold “almost everywhere” rather than at every point, but as
usual I will gloss over this distinction.) Similarly, for $\hat{\Theta}$ to be Hermitian
we require $\int F_1 \cdot \varepsilon^{-1} F_2 = \int (\varepsilon^{-1} F_1)^* \cdot F_2$ where $F = \nabla \times H$, so that we can move the $\varepsilon^{-1}$ over to the left side of the inner product, and thus $\varepsilon^{-1}(x)$
must be Hermitian at every $x$. From problem 1, this implies that $\varepsilon(x)$ is
also Hermitian. Finally, to get real eigenvalues we saw from above that
we must have $\mu$ positive definite ($\int H^* \cdot \mu H > 0$ for $H \neq 0$); since this
must be true for all $H$ then $\mu(x)$ at each point must be a positive-definite
$3 \times 3$ matrix (positive eigenvalues). Similarly, $\hat{\Theta}$ must be positive semi-
definite, which implies that $\varepsilon^{-1}(x)$ is positive semi-definite (non-negative
eigenvalues), but since it has to be invertible we must have $\varepsilon(x)$ positive
definite (zero eigenvalues would make it singular). To sum up, we must
have $\varepsilon(x)$ and $\mu(x)$ being positive-definite Hermitian matrices at every $x$.
(The proof for the E eigenproblem is identical.)

Actually, there are a couple other possibilities. In part (b), we showed
that if $\hat{B}$ is positive-definite it leads to real eigenvalues etc. The same
properties, however, hold if $\hat{B}$ is negative-definite, and if both $\hat{A}$ and $\hat{B}$
are negative-definite we still get real, positive eigenvalues. Thus, another
possibility is for $\varepsilon$ and $\mu$ to be Hermitian negative-definite matrices. (For
a scalar $\varepsilon < 0$ and $\mu < 0$, this leads to materials with a negative real
index of refraction $n = -\sqrt{\varepsilon \mu}$!) Furthermore, $\varepsilon$ and $\mu$ could both be anti-
Hermitian instead of Hermitian (i.e., $\varepsilon^\dagger = -\varepsilon$ and $\mu^\dagger = -\mu$), although I’m
not aware of any physical examples of this. More generally, for any com-
plex number $z$, if we replace $\varepsilon$ and $\mu$ by $ze$ and $\mu/z$, then $\omega$ is unchanged
(e.g. making $z = i$ gives anti-Hermitian matrices).