18.369 Problem Set 1

Due Wednesday, 10 February 2016.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of operators $\hat{O}$ by:
$$\langle H_1, \hat{O}H_2 \rangle = \langle \hat{O}^\dagger H_1, H_2 \rangle$$
for all $H_1$ and $H_2$ in the vector space. Show that for a finite-dimensional Hilbert space, where $H$ is a column vector $h_n$ ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}, H^{(2)} \rangle$ is the ordinary conjugated dot product
$$\sum_n h_n^{(1)} \bar{h}_n^{(2)}$$
the above adjoint definition corresponds to the conjugate-transpose for matrices. (Thus, as claimed in class, “swapping rows and columns” is the consequence of the “real” definition of transposition/adjoints, not the source.)

(b) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called unitary. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues $u$ of a unitary operator have unit magnitude ($|u| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(c) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

Problem 2: Maxwell eigenproblems

(a) As in class, assume $\varepsilon(x)$ real and positive (and that all function spaces are chosen so that the integrals you need exist etc.). In class, we eliminated $E$ from Maxwell’s equations to get an eigenproblem in $H$ alone, of the form $\hat{\Theta}H(x) = \frac{\omega^2}{c^2}H(x)$. Show that if you instead eliminate $H$, you cannot get a Hermitian eigenproblem in $E$ for the usual inner product $\langle E_1, E_2 \rangle = \int E_1^* \cdot E_2$ except for the trivial case $\varepsilon = \text{constant}$. Instead, show that you get a generalized Hermitian eigenproblem: an equation of the form $\hat{A}E(x) = \frac{\omega^2}{c^2} \hat{B}E(x)$, where both $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For any generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e. $\langle E, \hat{B}E \rangle > 0$ for all $E(x) \neq 0$), show that the eigenvalues (i.e., the solutions of $\hat{A}E = \lambda \hat{B}E$) are real and that different eigenfunctions $E_1$ and $E_2$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $E$ eigenproblem above was indeed positive definite.

(c) Alternatively, show that $\hat{B}^{-1}\hat{A}$ is Hermitian under a modified inner product $\langle E, E' \rangle_B = \langle E, \hat{B}E' \rangle$ for Hermitian $\hat{A}$ and $\hat{B}$ and positive-definite $\hat{B}$ with respect to the original $\langle E, E' \rangle$ inner product; the results from the previous part then follow.

(d) Show that both the $E$ and $H$ formulations lead to generalized Hermitian eigenproblems with real $\omega$ if we allow magnetic materials $\mu(x) \neq 1$ (but require $\mu$ real, positive, and independent of $H$ or $\omega$).

(e) $\mu$ and $\varepsilon$ are only ordinary numbers for isotropic media. More generally, they are $3 \times 3$ matrices (technically, rank 2 tensors)—thus, in an anisotropic medium, by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in $E$ (or $H$) with real eigenfrequencies $\omega$.

---

1Technically, we mean $u = 0$ “almost everywhere” (e.g. excluding isolated points).